EINDHOVEN UNIVERSITY OF TECHNOLOGY

Design of Freeform Imaging Systems: Mathematical Model and Numerics

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Sanjana Verma

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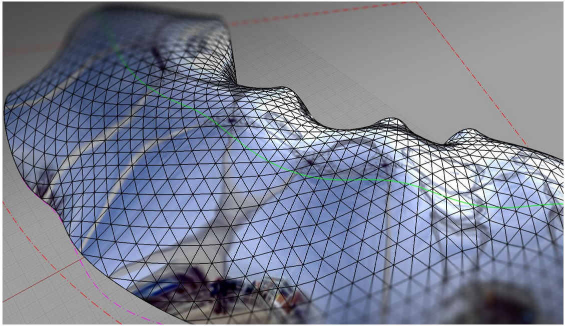

A freeform surface.

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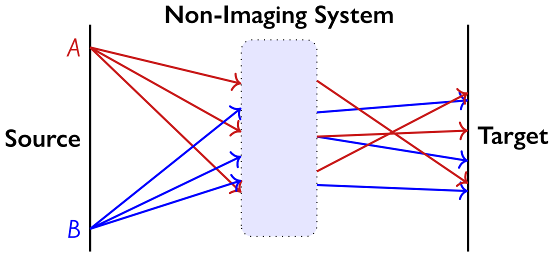
Non-Imaging vs. Imaging Optics



Goal: Transfer energy from source to target

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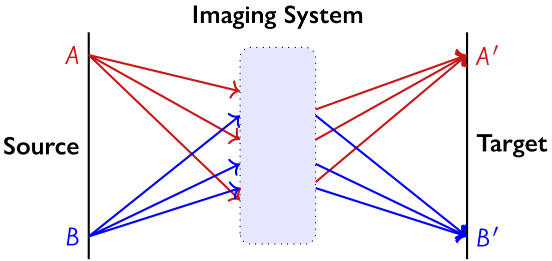
Nonimaging vs. Imaging Optics



Goal: Form a perfect image of the source in the target

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Aberrations

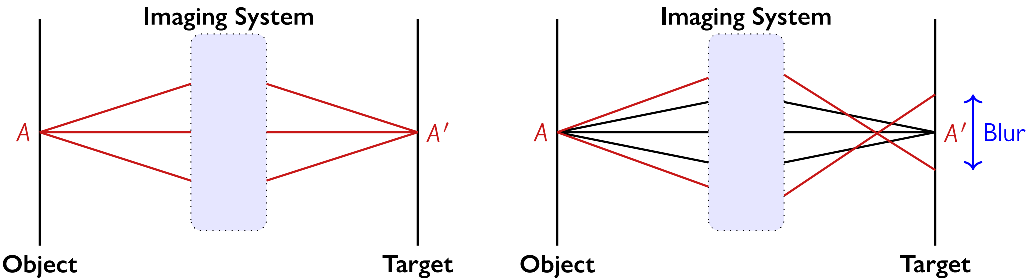
Aberrations are small deviations from a perfect image

Figure: A perfect image

Figure: An aberrated image

Aberrations

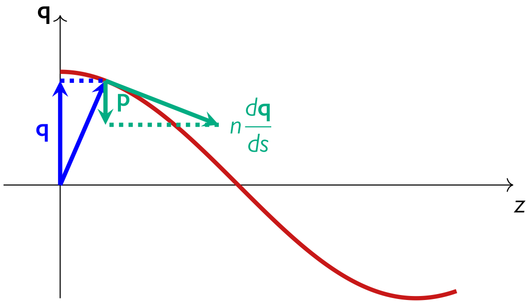
Aberrations are small deviations from a perfect image



(a) Ideal image formation (b) Image formation in reality

Goal: Minimize the spot size for all object points

Mathematical Description



A ray is fully characterized by phase-space variables

1. Position: $\boldsymbol{q} \in \mathbb{R}^2$
2. Momentum: $\boldsymbol{p} \in \mathbb{R}^2$

- Phase Space:

Source coordinates	$(\boldsymbol{q}_s, \boldsymbol{p}_s)$
Target coordinates	$(\boldsymbol{q}_t, \boldsymbol{p}_t)$

- Optical Map:

$$\begin{bmatrix} \boldsymbol{q}_t \\ \boldsymbol{p}_t \end{bmatrix} = \mathcal{M} \begin{bmatrix} \boldsymbol{q}_s \\ \boldsymbol{p}_s \end{bmatrix}$$

- Ideal Imaging: $\mathcal{M} : (\boldsymbol{q}_s, \boldsymbol{p}_s) \longrightarrow (\boldsymbol{q}_t, \boldsymbol{p}_t)$ is linear

$$\begin{bmatrix} \boldsymbol{q}_t \\ \boldsymbol{p}_t \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_s \\ \boldsymbol{p}_s \end{bmatrix}$$

A, B, C, and D are 2×2 constant matrices

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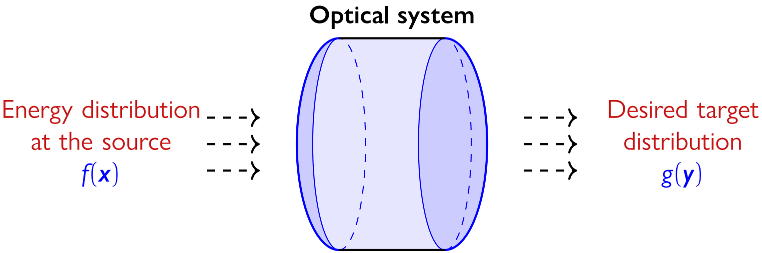
Inverse Methods in Nonimaging Optics



- **Goal:** Compute freeform optical elements
- **Optical Map:** $\boldsymbol{y} = \boldsymbol{m}(\boldsymbol{x})$
- **Energy Conservation:** $\int_A f(\boldsymbol{x})\,\mathrm{d}(\boldsymbol{x}) = \int_{m(A)} g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int_A g(\boldsymbol{m}(\boldsymbol{x}))\,|\det(D\boldsymbol{m}(\boldsymbol{x}))|\,\mathrm{d}\boldsymbol{x}$
- **Monge-Ampère Equation:** $\det(\mathrm{D}\boldsymbol{m}(\boldsymbol{x})) = \pm\frac{f(\boldsymbol{x})}{g(\boldsymbol{m}(\boldsymbol{x}))}$

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Aim

Design freeform reflectors for 3D imaging systems
using inverse methods from nonimaging optics

How?

Global energy conservation such that an imaging condition is satisfied

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Mathematical Model

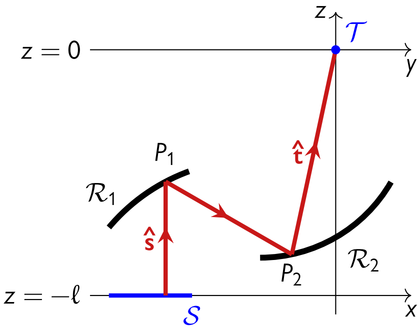
A Double-Reflector System

- **First Reflector**: $\mathcal{R}_1 : z = -\ell + u(\boldsymbol{x})$
- **Second Reflector**: $\mathcal{R}_2 : \mathbf{r} = -w(\hat{\mathbf{t}}) \cdot \hat{\mathbf{t}}$
- **Optical Path Length**: $V = u(\boldsymbol{x}) + |P_1P_2| + w(\hat{\mathbf{t}})$
- **Reduced Optical Path Length**: $\beta = V - \ell$
- **Cost Function**: $u_1(\boldsymbol{x}) + u_2(\boldsymbol{y}) = c(\boldsymbol{x}, \boldsymbol{y})$

$$u_1(\boldsymbol{x}) = \log\left(-u - \frac{|\boldsymbol{x}|^2}{2\beta} + \frac{V+\ell}{2}\right)$$

$$u_2(\boldsymbol{y}) = \log\left(\frac{1}{w} - \frac{2|\boldsymbol{y}|^2}{\beta\,(1+|\boldsymbol{y}|^2)}\right)$$

$$c(\boldsymbol{x}, \boldsymbol{y}) = \log\left(\frac{1}{1+|\boldsymbol{y}|^2}\left(1 + \frac{2\boldsymbol{x}\cdot\boldsymbol{y}}{\beta} + \frac{|\boldsymbol{x}|^2|\boldsymbol{y}|^2}{\beta^2}\right)\right)$$



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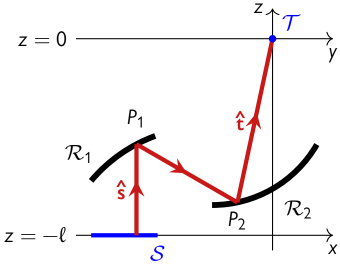
- **Energy Conservation:**
  For any subset $\mathcal{A} \subseteq Q_s \times P_s$ and image set $\mathcal{M}(\mathcal{A}) \subseteq Q_t \times P_t$

$$\iint_{\mathcal{A}} L_s(\boldsymbol{q}_s, \boldsymbol{p}_s)\,\mathrm{d}\boldsymbol{q}_s\,\mathrm{d}\boldsymbol{p}_s = \iint_{\mathcal{M}(\mathcal{A})} L_t(\boldsymbol{q}_t, \boldsymbol{p}_t)\,\mathrm{d}\boldsymbol{q}_t\,\mathrm{d}\boldsymbol{p}_t$$

- $L_s(\boldsymbol{q}_s, \boldsymbol{p}_s) = f(\boldsymbol{q}_s)\delta(\boldsymbol{p}_s)$ and $L_t(\boldsymbol{q}_t, \boldsymbol{p}_t) = g(\boldsymbol{p}_t)\delta(\boldsymbol{q}_t)$



  - Consider the mapping $\tilde{\boldsymbol{m}}(\boldsymbol{q}_s) = \boldsymbol{p}_t$. Then, for any subset $\tilde{\mathcal{A}} \subseteq Q_s$ and image set $\tilde{\boldsymbol{m}}(\tilde{\mathcal{A}}) \subseteq P_t$

$$\int_{\tilde{\mathcal{A}}} f(\boldsymbol{q}_s)\,\mathrm{d}\boldsymbol{q}_s = \int_{\tilde{\boldsymbol{m}}(\tilde{\mathcal{A}})} g(\boldsymbol{p}_t)\,\mathrm{d}\boldsymbol{p}_t$$

  - Substitute the mapping $\tilde{\boldsymbol{m}}$

$$\det(D\tilde{\boldsymbol{m}}(\boldsymbol{q}_s)) = \pm\frac{f(\boldsymbol{q}_s)}{g(\tilde{\boldsymbol{m}}(\boldsymbol{q}_s))}, \quad \text{Transport Boundary Condition: } \tilde{\boldsymbol{m}}(\partial Q_s) = \partial P_t$$

- **Optical Map for Ideal Imaging**:
  $\mathcal{M} : (\boldsymbol{q}_\mathrm{s}, \boldsymbol{p}_\mathrm{s}) \to (\boldsymbol{q}_\mathrm{t}, \boldsymbol{p}_\mathrm{t})$ is a linear map

$$\begin{bmatrix} \boldsymbol{q}_\mathrm{t} \\ \boldsymbol{p}_\mathrm{t} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_\mathrm{s} \\ \boldsymbol{p}_\mathrm{s} \end{bmatrix}$$

- $\mathcal{M}$ is symplectic, i.e.,
  $\det(\mathcal{M}) = |\partial(\boldsymbol{q}_\mathrm{t}, \boldsymbol{p}_\mathrm{t}) / \partial(\boldsymbol{q}_\mathrm{s}, \boldsymbol{p}_\mathrm{s})| = 1$
- Using $\boldsymbol{p}_\mathrm{s} = \mathbf{0}$ and $\boldsymbol{q}_\mathrm{t} = \mathbf{0}$

$$\tilde{\boldsymbol{m}}(\boldsymbol{q}_\mathrm{s}) = \boldsymbol{p}_\mathrm{t}, \quad \boldsymbol{p}_\mathrm{t} = C\boldsymbol{q}_\mathrm{s}$$

$$\det(C) = \det(B)^{-1} = \det(\mathrm{D}\tilde{\boldsymbol{m}}(\boldsymbol{q}_\mathrm{s}))$$

- **Condition for Rotational Symmetry:**

$$C = \pm k I_{2\times 2}$$

$$\det(C) = k^2$$

- Using energy conservation and rotational symmetry

$$\det(\mathrm{D}\tilde{\boldsymbol{m}}(\boldsymbol{q}_\mathrm{s})) = \frac{f(\boldsymbol{q}_\mathrm{s})}{g(\tilde{\boldsymbol{m}}(\boldsymbol{q}_\mathrm{s}))} = k^2$$

- Optical map:

$$\tilde{\boldsymbol{m}}(\boldsymbol{q}_\mathrm{s}) = \pm k \boldsymbol{q}_\mathrm{s}$$

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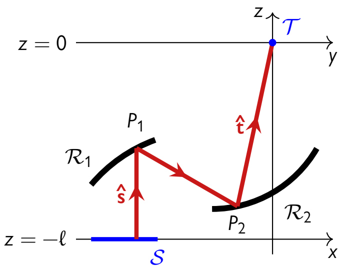
$\tilde{\boldsymbol{m}} : Q_s \to P_t$	$\det(D\tilde{\boldsymbol{m}}(\boldsymbol{q}_s)) = \dfrac{f(\boldsymbol{q}_s)}{g(\tilde{\boldsymbol{m}}(\boldsymbol{q}_s))}$
$\boldsymbol{m} : \mathcal{S} \to \mathcal{T}$	$\det(D\boldsymbol{m}(\boldsymbol{x})) = \dfrac{f(\boldsymbol{x})}{g(\boldsymbol{m}(\boldsymbol{x}))} \dfrac{(1+\lvert\boldsymbol{m}(\boldsymbol{x})\rvert^2)^3}{4(1-(\lvert\boldsymbol{m}(\boldsymbol{x})\rvert^2))}$



- **Optimal Transport Formulation:** The transport relation $u_1(\boldsymbol{x}) + u_2(\boldsymbol{y}) = c(\boldsymbol{x}, \boldsymbol{y})$ has infinitely many solutions, but we restrict ourselves to a c-convex pair

$$u_1(\boldsymbol{x}) = \max_{\boldsymbol{y} \in \mathcal{T}} [c(\boldsymbol{x}, \boldsymbol{y}) - u_2(\boldsymbol{y})] \qquad \forall \boldsymbol{x} \in \mathcal{S}$$

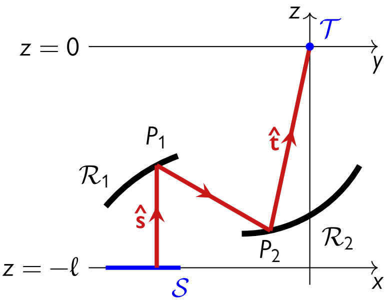
$$u_2(\boldsymbol{y}) = \max_{\boldsymbol{x} \in \mathcal{S}} [c(\boldsymbol{x}, \boldsymbol{y}) - u_1(\boldsymbol{x})] \qquad \forall \boldsymbol{y} \in \mathcal{T}$$

- **Transformation to Polar Coordinates:** $\boldsymbol{m}^* : \omega \to \mathcal{T}, \quad \boldsymbol{m}^*(\zeta) = \boldsymbol{y}$, where $\boldsymbol{x} = \boldsymbol{x}(\zeta), \zeta = (r, \vartheta)$
- The transport relation $\tilde{u}_1(\zeta) + u_2(\boldsymbol{y}) = \tilde{c}(\zeta, \boldsymbol{y})$ has infinitely many solutions, but we restrict ourselves to a c-convex pair

$$\tilde{u}_1(\zeta) = \max_{\boldsymbol{y}\in\mathcal{T}}[\tilde{c}(\zeta, \boldsymbol{y}) - u_2(\boldsymbol{y})] \qquad \forall \zeta \in \omega$$

$$u_2(\boldsymbol{y}) = \max_{\zeta\in\omega}[\tilde{c}(\zeta, \boldsymbol{y}) - \tilde{u}_1(\zeta)] \qquad \forall \boldsymbol{y} \in \mathcal{T}$$

- **First Reflector:**
  Differentiate $\tilde{u}_1(\zeta) + u_2(\boldsymbol{y}) = \tilde{c}(\zeta, \boldsymbol{y})$ and substitute $\boldsymbol{y} = \boldsymbol{m}^*(\zeta)$
  $$\frac{\partial \tilde{u}_1}{\partial r} = \frac{\pm 2kr}{\pm kr^2 + \beta\left(1 + \sqrt{1 - (kr)^2}\right)}$$
  Solve ODE for u_1 and find $u(\zeta)$
- **Second Reflector:**
  We have $u_2(\boldsymbol{m}^*(\zeta)) = \tilde{c}(\zeta, \boldsymbol{m}^*(\zeta)) - \tilde{u}_1(\zeta)$
  Compute u_2 and subsequently w



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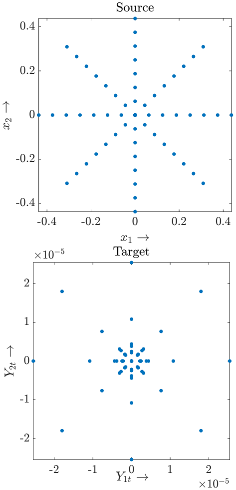
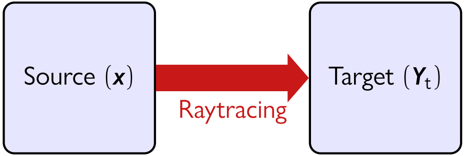
Verification with Raytracing

Quantifying Aberrations



Scatter plots



Size of the spot = RMS spot size = $\left(\mathrm{Var}[\boldsymbol{Y}_\mathrm{t}]\right)^{1/2}$

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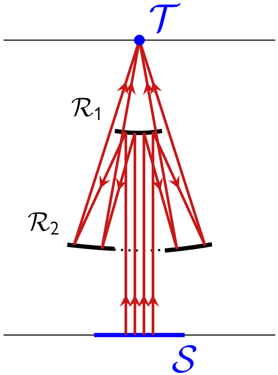
Calculating Inverse Freeform Imaging Reflectors

For an imaging system: $\frac{f}{g} = k^2$

What is the value of k^2?

1. A Schwarzschild telescope minimizes aberrations
2. Raytrace ω, and find the corresponding $\mathcal{T}$
3. With global energy conservation, find $f/g = k^2$
4. Compute the optical map $\boldsymbol{m}^* : \omega \to \mathcal{T}$
5. Compute the inverse freeform reflectors



A Schwarzschild telescope

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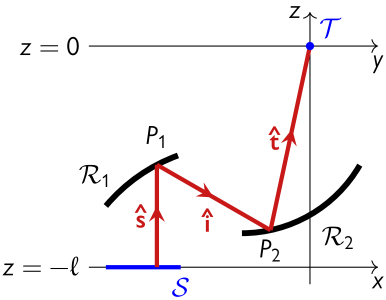
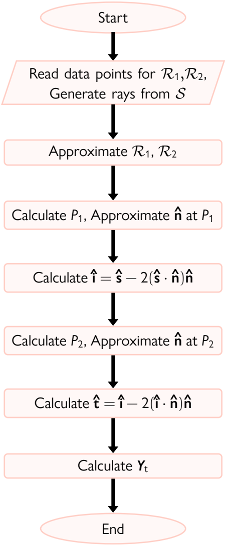
Raytracing



Requirement: Use a coarse grid to approximate reflectors and their partial derivatives as smooth functions



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B-Spline Quasi-Interpolation for Raytracing

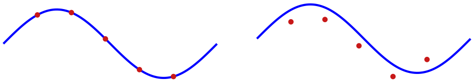
- **Quasi-Interpolants:** For a real-valued function $f \in \mathcal{B}_d$, where the $\mathcal{B}_d$ is the bivariate space of polynomials of total degree at most d,

$$Q_d[f](x,y) = \sum_{s=1}^{n} \sum_{r=1}^{n} \lambda_{rs}(f) B_{r,d,\boldsymbol{\xi}}(x) B_{s,d,\boldsymbol{\mu}}(y),$$

  where

  1. $B_{r,d,\boldsymbol{\xi}}(x)$ is the r^{th} B-spline of degree d defined on an open knot sequence $\boldsymbol{\xi} = \{\xi_r\}_{r=1}^{n+d+1}$ in the x-direction
  2. $B_{s,d,\boldsymbol{\mu}}(y)$ is the s^{th} B-spline of degree d defined on an open knot sequence $\boldsymbol{\mu} = \{\mu_s\}_{s=1}^{n+d+1}$ in the y-direction
  3. n is the total number of B-splines
  4. $\lambda_{rs}(f)$ are coefficients depending on known values of f

- **Partial Derivatives:**
  $D_x[f](x,y) = \sum_{s=1}^{n} \sum_{r=2}^{n} c_{rs}(f) B_{r,d-1,\boldsymbol{\xi}}(x) B_{s,d,\boldsymbol{\mu}}(y)$
  $D_y[f](x,y) = \sum_{s=2}^{n} \sum_{r=1}^{n} c_{rs}(f) B_{r,d,\boldsymbol{\xi}}(x) B_{s,d-1,\boldsymbol{\mu}}(y)$



(a) B-Spline interpolation (b) Quasi-interpolation

Advantages

- **Local Method:** $\lambda_{rs}(f)$ are linear combinations of the values of f at points in the neighborhood of the support of the B-splines
- Does not depend on huge data sets
- **Direct Method:** Built directly without solving any systems of linear equations
- Low computational cost
- Flexibility and simplicity of constructing tailor-made approximation schemes

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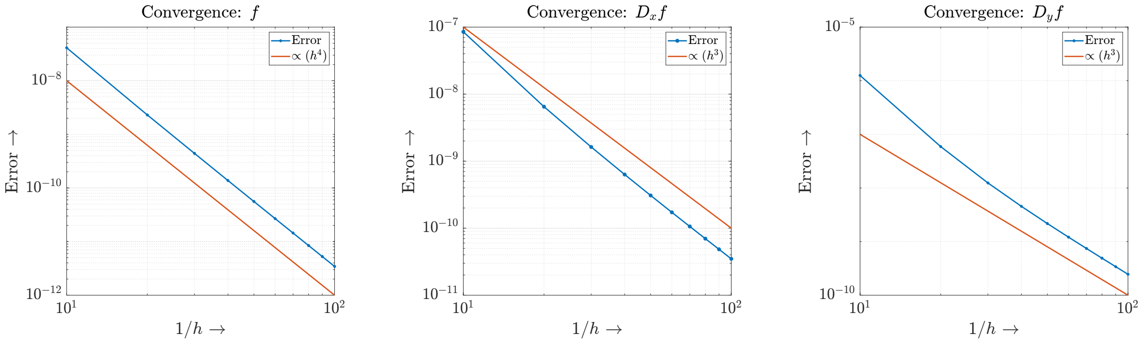
- **Approximation Order:** Let $h = \max(h_x, h_y)$, where $h_x = \max(\xi_{r+1} - \xi_r)$, $h_y = \max(\mu_{s+1} - \mu_s)$
  1. Function: $\|f - f_0\| = \mathcal{O}(h^{d+1})$
  2. First-order derivative: $\|Df - Df_0\| = \mathcal{O}(h^d)$



Approximate $f(x, y) = \cos(xy) + \sinh(x) + \exp(xy)$, $(x, y) \in [0, 0.5] \times [0, 0.5]$ using $d = 3$, $B_{r,3,\xi}(x)$, $B_{s,3,\mu}(y)$

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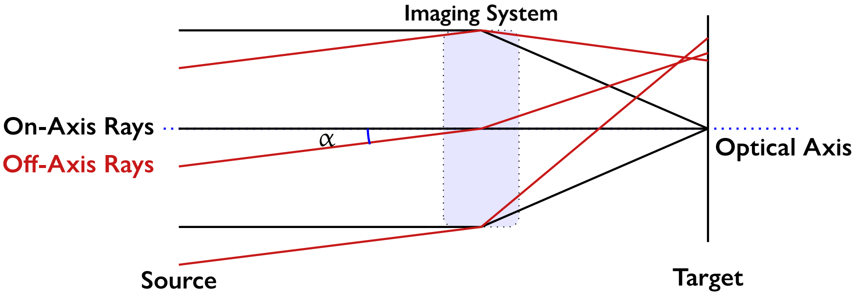
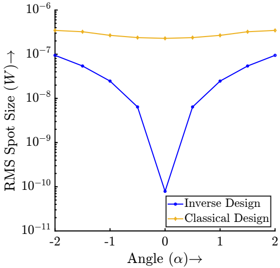
Numerical Results



Goal: Compare RMS spot sizes of on-axis and off-axis rays for classical and inverse designs

Numerical Results



Comparison of RMS spot sizes

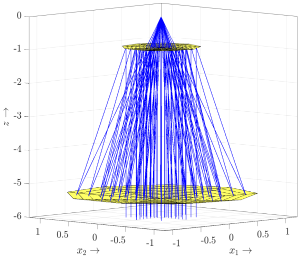
Table: RMS spot sizes for different angles

Angle	$0°$	$\pm 2°$
Inverse Design	7.417e − 11	9.075e − 08
Classical Design	1.386e − 07	3.461e − 07

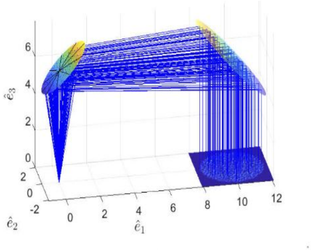
Spot sizes for inverse freeform design are smaller than the classical design for both on-axis and off-axis rays

Concluding Remarks

- Inverse methods for the design of nonimaging systems can be adapted for imaging systems by choosing suitable energy distributions
- Inverse freeform design is superior to the classical design
- Inverse design can be a good starting point for optimization
- Extend the method to folded optical systems



(a) On-axis system



(b) Folded system

https://martijna.win.tue.nl/Optics/

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