Numerical algorithms for the reconstruction of space-dependent sources in thermoelasticity

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Joint work with F. Maes

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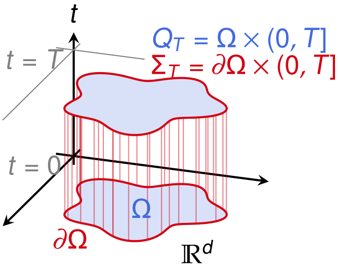
Outline

Inverse spatial source problems: Heat equation

- Heat problem on a Lipschitz domain $\Omega \subset \mathbb{R}^d$, with final time $T > 0$
- Separated-variable heat source: $h \in \mathrm{L}^2(0,T)$ and $f \in \mathrm{L}^2(\Omega)$

Problem formulation:



- $$\begin{cases} \partial_t u - \kappa \Delta u = h(t) f(x) & \text{in } Q_T \\ u(x,t) = 0 & x \in \Sigma_T \\ u(x,0) = u_0(x) & x \in \Omega \end{cases}$$

DP: Given the source hf, determine $u_T(x) := u(x,T)$

- If $\kappa > 0$, $h \in \mathrm{L}^2(0,T)$, $f \in \mathrm{L}^2(\Omega)$ and $u_0 \in \mathrm{H}_0^1(\Omega)$, then:
  - A unique weak solution exists:

$$u \in \mathrm{C}\left([0,T], \mathrm{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0,T), \mathrm{H}_0^1(\Omega)\right), \quad \partial_t u \in \mathrm{L}^2\left((0,T), \mathrm{L}^2(\Omega)\right)$$

  - The final state $u(\cdot, T) \in \mathrm{L}^2(\Omega)$ is well-defined
  - The measurement operator $M_T : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ is well-defined:

$$f \overset{M_T}{\longmapsto} u_T$$

Inverse spatial source problems: Heat equation

- Measurement operator M_T:

$$\mathrm{L}^2(\Omega) \ni f \xmapsto{M_T} u_T \in \mathrm{L}^2(\Omega)$$

IP: Given u_T, determine f such that $M_T(f) = u_T$

- If $u_0 \in \mathrm{H}_0^1(\Omega)$, one can show that for the solution u to the DP:

$$\int_0^T \|\partial_t u(\cdot,t)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + \max_{t\in[0,T]} \|\nabla u(\cdot,t)\|^2_{\mathbf{L}^2(\Omega)} \leq C\,\|f\|^2_{\mathrm{L}^2(\Omega)} \qquad (\bigstar_{\mathrm{EE}})$$

for some constant $C = C(T,\kappa,\|h\|_{\mathrm{L}^2(0,T)},\|u_0\|_{\mathrm{H}_0^1(\Omega)}) > 0$

- M_T is linear if $u_0 = 0$ (achieved by linear superposition)
- Consequences of $(\bigstar_{\mathrm{EE}})$:
  - M_T is bounded
  - $M_T(f) \in \mathrm{H}_0^1(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}^2(\Omega)$
  - So M_T is compact
  - ✓ Well-posed DP, ✗ Ill-posed IP

References: [Cannon, 1968], [Rundell, 1980], [Prilepko and Solov'ev, 1987], [Solov'ev, 1989], [Isakov, 1990], [Johansson and Lesnic, 2007], [Erdem et al., 2013], [Slodička and Johansson, 2016], [Slodička, 2020]

Inverse spatial source problems: Heat equation – Uniqueness

IP: Given $u_T = u(\cdot, T)$, determine f such that $M_T(f) = u_T$

- Uniqueness: consider the homogeneous problem:

$$\begin{cases} \partial_t u - \kappa \Delta u &= hf & \text{in } Q_T \\ u(x,t) &= 0 & \text{on } \Sigma_T \\ u(x,0) &= 0 = u_T(x) & \text{in } \Omega \end{cases}$$

- $(\cdot,\cdot)$ denotes inner product in $\mathrm{L}^2(\Omega)$ or $\mathbf{L}^2(\Omega)$
- Corresponding variational form (for a.a. $t \in (0,T)$):

$$(\partial_t u(t), \varphi) + \kappa (\nabla u(t), \nabla \varphi) = h(t)\,(f, \varphi), \quad \forall \varphi \in \mathrm{H}_0^1(\Omega) \qquad (\bigstar_{\mathrm{VFDP}})$$

- Observe that

$$\int_0^T (f, \partial_t u(\cdot,t))\, \mathrm{d}t = (f, u_T - u_0) = 0$$

Inverse spatial source problems: Uniqueness – continued

- Suppose $h \in C^1([0,T])$ with $h > 0$ and $h' \geq 0$
- Choose $\varphi = \partial_t u(\cdot,t) \in H_0^1(\Omega)$ in ($\bigstar_{\mathrm{VFDP}}$) and rewrite:

$$\underbrace{\int_0^T \frac{1}{h(t)} \|\partial_t u(\cdot,t)\|^2 \,\mathrm{d}t}_{\geq 0} + \frac{\kappa}{2}\int_0^T \frac{1}{h(t)} \partial_t \|\nabla u(\cdot,t)\|^2 \,\mathrm{d}t = 0$$

- Apply partial integration on the second term:

$$\int_0^T \frac{1}{h(t)} \partial_t \|\nabla u(\cdot,t)\|^2 \,\mathrm{d}t = \left[\frac{1}{h(t)} \|\nabla u(\cdot,t)\|^2\right]_{t=0}^{t=T} + \int_0^T \frac{h'(t)}{h(t)^2} \|\nabla u(\cdot,t)\|^2 \,\mathrm{d}t \geq 0$$

- Hence, $u \equiv 0$ a.e. in Q_T, and therefore $f \equiv 0$ a.e. in Ω $\rightsquigarrow$ solution to IP is unique
- See [Isakov, 1990, Slodička and Johansson, 2016]

Remark: Also uniqueness for $\{h < 0, h' \leq 0\}$ or $\{h \neq 0, \left(\frac{h'}{h}\right)' \leq 0\}$ for $h \in C^2([0,T])$

IP requires higher regularity compared to DP!

Inverse Spatial Source Problems: Remarks

Parabolic equation:

✓ If $h > 0$, $h' \geq 0$ or $h < 0$, $h' \leq 0$:

- Uniqueness holds from final data u_T
- Also from **time-averaged data**:

$$\Psi_T(\cdot) := \int_0^T u(\cdot, t)\,\mathrm{d}t$$

Wave equation:

- Equation: $\partial_{tt} u - c^2 \Delta u = h(t) f(x)$

✓ [Isakov, 1990, Ch. 7]: Uniqueness from u_T if:

$$h(t) \gtrless 0 \text{ and } h'(t) \gtrless 0 \quad \forall t \in [0, T]$$

✗ No uniqueness if $h \equiv 1$

- But uniqueness holds in presence of damping

✓ Uniqueness from Ψ_T if $h \neq 0$

- **Next: Reconstruction methods** v

H.W. Engl, M. Hanke, A. Neubauer Regularization of Inverse Problems *Mathematics and its Applications.* 1996

Reconstruction of f – Landweber method

- Landweber scheme to reconstruct f from u_T:
  - choose $0 < \alpha < \|M_T\|^{-2}_{\mathcal{L}(\mathrm{L}^2(\Omega),\mathrm{L}^2(\Omega))}$
  - guess $f_0 \in \mathrm{L}^2(\Omega)$
  - iterate

$$f_k = f_{k-1} - \alpha M_T\left(M_T(f_{k-1}) - u_T\right), \quad k = 1, 2, \ldots$$

- Since $f_k - f = (\mathbb{I} - \alpha M_T^2)(f_{k-1} - f)$, convergence of $f_k \to f$ in $\mathrm{L}^2(\Omega)$ follows
- Estimate ($\bigstar_{\mathrm{EE}}$) implies $u_k \to u$ in $C\left([0, T], \mathrm{L}^2(\Omega)\right)$
- For noisy data: apply Morozov's discrepancy principle as stopping criterion

✓	✗
only forward solver	slow
'easy' to implement	$f_k\|_{\partial\Omega} \equiv f_0\|_{\partial\Omega}$ for all k
	Choice of α?

Reconstruction of f – Tikhonov minimisation

- Tikhonov functional $\mathcal{J} : \mathrm{L}^2(\Omega) \to \mathbb{R}$ quantifies the misfit:

$$\mathcal{J}(f) = \frac{1}{2} \|M_T(f) - u_T\|^2_{\mathrm{L}^2(\Omega)}$$

- Goal: find f such that the model output $M_T(f)$ fits the data u_T:

$$\operatorname{argmin}_{f \in \mathrm{L}^2(\Omega)} \mathcal{J}(f)$$

- How to find this minimiser?
  - Compute Gâteaux derivative $\mathcal{J}'(f; \tilde{f})$
  - Use an adjoint PDE to express the gradient $\nabla \mathcal{J}[f]$

Reconstruction of f – Tikhonov minimisation

Directional derivative of the forward map $f \mapsto u$

- Let $u(\cdot,\cdot;f)$ be the solution to DP with source term f
- Consider the directional derivative δu in direction $\tilde{f} \in \mathcal{H} \subseteq \mathrm{L}^2(\Omega)$:

$$\delta u(\cdot,\cdot;f,\tilde{f}) = \lim_{\varepsilon\to 0} \frac{u(\cdot,\cdot;f+\varepsilon\tilde{f}) - u(\cdot,\cdot;f)}{\varepsilon}$$

- Then δu satisfies the sensitivity problem:

$$\begin{cases} \partial_t(\delta u) - \kappa\Delta(\delta u) &= h(t)\,\tilde{f} & \text{in } Q_T \\ \delta u &= 0 & \text{on } \Sigma_T \\ \delta u(x,0) &= 0 & \text{in } \Omega \end{cases}$$

- Variational formulation (for a.a. $t \in (0,T)$):

$$(\partial_t(\delta u)(t),\varphi) + \kappa\,(\nabla(\delta u)(t),\nabla\varphi) = h(t)\left(\tilde{f},\varphi\right), \quad \forall\varphi \in H_0^1(\Omega) \qquad (\bigstar_{\mathrm{VFSP}})$$

- Key observation:

$$(\delta u)(T) = M_T(\tilde{f})$$

Reconstruction of f – Tikhonov minimisation

Gateaux derivative of $\mathcal{J}$ and gradient in $L^2(\Omega)$

- Compute the Gâteaux derivative of $\mathcal{J}$ at f in the direction $\tilde{f}$:

$$\mathcal{J}'(f;\tilde{f}) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[\mathcal{J}(f+\varepsilon\tilde{f}) - \mathcal{J}(f)\right] = \cdots = ((\delta u)(T), u(T;f) - u_T) = \left(M_T(\tilde{f}), M_T(f) - u_T\right)$$

- The map $\mathcal{J}'(f;\cdot) : L^2(\Omega) \to \mathbb{R}$ is a continuous linear functional
- By the Riesz representation theorem, there exists a unique element $\nabla\mathcal{J}[f] \in L^2(\Omega)$ such that

$$\mathcal{J}'(f;\tilde{f}) = \left(\nabla\mathcal{J}[f], \tilde{f}\right), \quad \forall \tilde{f} \in L^2(\Omega)$$

- Which element? $\Rightarrow$ Use an "adjoint problem" to characterise $\nabla\mathcal{J}[f]$!

Reconstruction of f – Tikhonov minimisation

Adjoint problem and gradient of $\mathcal{J}$

- Take $\varphi = u^* \in \mathrm{H}^1_0(\Omega)$ (i.e. $u^*|_{\partial\Omega} = 0$) in sensitivity problem ($\bigstar_{\mathrm{VFSP}}$)
- Apply partial integration (in time) and Green's theorem (in space)
- As $(\delta u)(0) = 0$ and $(\delta u)|_{\partial\Omega} = 0$:

$$\int_0^T \int_\Omega \delta u \cdot [-\partial_t u^* - \kappa \Delta u^*] \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u^*(\cdot, T) \cdot M_T(\tilde{f}) \,\mathrm{d}x = \int_\Omega \tilde{f} \cdot \left[\int_0^T h u^* \,\mathrm{d}t \right] \mathrm{d}x$$

- Hence

$$\left(\nabla \mathcal{J}[f], \tilde{f} \right) = \left(M_T(\tilde{f}), M_T(f) - u_T \right) = \left(\int_0^T h(t) u^*(t) \,\mathrm{d}t, \tilde{f} \right),$$

$$\boxed{\nabla \mathcal{J}[f] = \int_0^T h(t)\, u^*(t) \,\mathrm{d}t \in \mathrm{L}^2(\Omega)}$$

where u^* solve the adjoint problem

$$\begin{cases} -\partial_t u^* - \kappa \Delta u^* = 0 & \text{in } Q_T \\ u^*(x,t) = 0 & x \in \Sigma_T \\ u^*(x,T) = M_T(f) - u_T & x \in \Omega \end{cases}$$

Remark: Expression for gradient of $\mathcal{J}$ is not unique: taking $\varphi = \partial_t u^*$ leads to
$\nabla \mathcal{J}[f] = \int_0^T h(t) \partial_t u^*(t) \,\mathrm{d}t$ with

$$\begin{cases} -\partial_t u^* - \kappa \Delta u^* = u_T - M_T(f) & \text{in } Q_T \\ u^*(x,t) = 0 & x \in \Sigma_T \\ u^*(x,T) = 0 & x \in \Omega \end{cases}$$

- [Johansson and Lesnic, 2007]: $\mathcal{J}$ is strictly convex $\rightsquigarrow$ unique minimiser

Reconstruction of f – Tikhonov minimisation

- Use the (standard) steepest descent algorithm to minimise $\mathcal{J}$:
  - Initialise with a guess $f_0 \in L^2(\Omega)$ and compute $\nabla\mathcal{J}[f_0]$
  - Update iteratively via

$$\begin{cases} f_n = f_{n-1} - \tau_n \nabla\mathcal{J}[f_{n-1}] \\ \tau_n = \operatorname{argmin}_{\tau>0} \mathcal{J}\left(f_{n-1} - \tau\nabla\mathcal{J}[f_{n-1}]\right) \end{cases}$$

- Optimal stepsize $\tau_n > 0$ from a 1D minimisation problem:

$$\tau_n = \frac{\left(M_T(f_{n-1}) - u_T, M_T(\nabla\mathcal{J}[f_{n-1}])\right)}{\|M_T(\nabla\mathcal{J}[f_{n-1}])\|^2}$$

- Combined with stopping criterion: $J(f_{n+1}) > J(f_n)$

✓	✗
(very) quick forward & adjoint solver	$f_k\|_{\partial\Omega} \equiv f_0\|_{\partial\Omega}$ for all k

- Alternative: conjugate gradient method

Reconstruction of f – Sobolev Gradient Method

- Motivation: How can we improve updates near the boundary?
- Idea: Add a post-processing step to compute a smoother gradient
- Given a guess $f_0 \in \mathrm{L}^2(\Omega)$, iterate as follows:
  - Compute the standard $\mathrm{L}^2(\Omega)$-gradient $\nabla\mathcal{J}[f_{n-1}]$ from the adjoint problem
  - Compute the Sobolev gradient $\nabla_S\mathcal{J}[f_{n-1}]$ as the weak solution $\mathcal{K}$ to:

$$\begin{cases} -\nabla\cdot(r_1\nabla\mathcal{K}) + r_0\mathcal{K} = \nabla\mathcal{J}[f_{n-1}] & \text{in } \Omega \\ \nabla\mathcal{K}\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega \end{cases}$$

    with weights $r_0, r_1 : \Omega \to \mathbb{R}$ satisfying: $0 < \underline{r_i} \le r_i(x) \le \overline{r_i}, i = 0, 1$
  - Update f_n:

$$\begin{cases} f_n = f_{n-1} - \tau_n\nabla_S\mathcal{J}[f_{n-1}] \\ \tau_n = \operatorname{argmin}_{\tau>0}\mathcal{J}\left(f_{n-1} - \tau\nabla_S\mathcal{J}[f_{n-1}]\right) \end{cases}$$

$\rightsquigarrow$ This enables updates *at the boundary* of Ω!

✓	✗
(very) quick updates at the boundary	Choice of r_0, r_1? Requires solving additional PDE

Identifiability of space-dependent sources in thermoelasticity

- System of type-III

$$\begin{cases} \rho\partial_{tt}\boldsymbol{u} - \mu\Delta\boldsymbol{u} - (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) + \gamma\nabla\theta &= g_{\boldsymbol{u}}(t)\boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{x}) & \text{in } Q_T \\ \rho C_s\partial_t\theta - \kappa\Delta\theta - \int_0^t k(t-s)\Delta\theta(s)\,\mathrm{d}s + T_0\gamma\nabla\cdot\partial_t\boldsymbol{u} &= g_\theta(t)f_\theta(\boldsymbol{x}) & \text{in } Q_T \\ \boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{0},\ \theta(\boldsymbol{x},t) &= 0 & \text{in } \Sigma_T \\ \boldsymbol{u}(\boldsymbol{x},0) = \overline{\boldsymbol{u}}_0(\boldsymbol{x}),\ \partial_t\boldsymbol{u}(\boldsymbol{x},0) = \overline{\boldsymbol{u}}_1(\boldsymbol{x}),\ \theta(\boldsymbol{x},0) &= \overline{\theta}_0(\boldsymbol{x}) & \text{in } \Omega \end{cases}$$

ISP	unknown	measurement	data	conditions
ISP1.1	$\boldsymbol{f_u}$	ξ_T	$g_{\boldsymbol{u}}, \boldsymbol{r}$	$g_{\boldsymbol{u}} \in C_1$, (C:PDE@$t=0$)
ISP1.2	$\boldsymbol{f_u}$	χ_T	$g_{\boldsymbol{u}}, \boldsymbol{r}$	$g_{\boldsymbol{u}} \in C_0$, (C)
ISP2	f_θ	ψ_T	g_θ, s	$g_\theta \in C_{1,*}$, (C)

- No damping term needed

$$\xi_T = \boldsymbol{u}(\cdot,T) \qquad \psi_T = \int_0^T \theta(\cdot,t)\,\mathrm{d}t$$

$$\chi_T = \int_0^T \boldsymbol{u}(\cdot,t)\,\mathrm{d}t$$

$$C_0 = \{g \in \mathrm{C}([0,T]) \mid 0 \notin g([0,T])\}$$

$$C_1 = \left\{g \in \mathrm{C}^1([0,T]) \mid \partial_t(h(t)^2) > 0,\ \forall t \in [0,T]\right\}$$

$$C_{1,*} = \left\{g \in \mathrm{C}^1([0,T]) \cap C_0 \mid \partial_t(h(t)^2) \geq 0,\ \forall t \in [0,T]\right\}$$

- Uniqueness for all ISPs

F. Maes, K. Van Bockstal Uniqueness for inverse source problems of determining a space-dependent source in thermoelastic systems. *J. Inverse Ill-Posed Probl.* ***30****(6), 845–856* (2022)

Reconstruction for inverse source problems in thermoelasticity

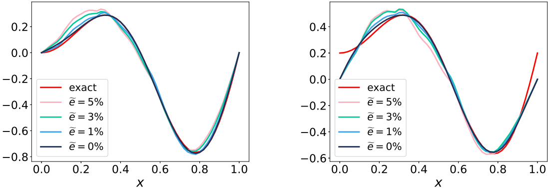
- Landweber method for **ISP1.2** with $N_T : \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega) : \boldsymbol{f} \mapsto \int_0^T \boldsymbol{u}(\cdot, t; \boldsymbol{f})\, \mathrm{d}t$
  - $\boldsymbol{f}_k = \boldsymbol{f}_{k-1} - \alpha N_T(N_T(\boldsymbol{f}_{k-1}) - \boldsymbol{X}_T)$, with $0 < \alpha < \|N_T\|^{-2}_{\mathcal{L}(\mathbf{L}^2(\Omega), \mathbf{L}^2(\Omega))}$

✓ $\boldsymbol{f}_k \to \boldsymbol{f}$ in $\mathbf{L}^2(\Omega)$ for any $\boldsymbol{f}_0 \in \mathbf{L}^2(\Omega)$

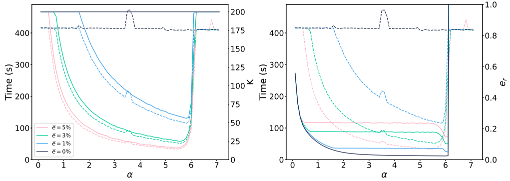
✓ only forward solver needed

✗ slow, zero update at the boundary, α

- FEniCS project
  The finite element library DOLFINx v0.8.0 [Logg and Wells, 2010, Logg et al., 2012b] from the FEniCS project [Logg et al., 2012a] is used
- Processor: 12th Gen Intel(R) Core(TM) i7-1270P @ 2.20 GHz with 32 GB RAM on Windows 11 Pro with Ubuntu (WSL) v22.04.4 LTS
- Max. number of iterations: 200



$f(x) = x\sin(2\pi x)$, $\alpha = 6$ $f(x) = x\sin(2\pi x) + 0.2$, $\alpha = 6.1$



K: number of iterations; e_r: relative error (solid lines)

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Reconstruction for inverse source problems in thermoelasticity

- Tikhonov minimisation for **ISP1.2**

$$\mathcal{I}_\beta(\boldsymbol{f}) = \frac{1}{2}\|N_T(\boldsymbol{f}) - \boldsymbol{X}_T\|^2_{\mathbf{L}^2(\Omega)} + \frac{\beta}{2}\|\boldsymbol{f}\|^2_{\mathbf{L}^2(\Omega)},\ \beta \geq 0$$

$$\nabla\mathcal{I}_\beta[\boldsymbol{f}] = \int_0^T h(t)\,\boldsymbol{u}^*(\cdot, t; \boldsymbol{f})\,\mathrm{d}t + \beta\boldsymbol{f}$$

$$\begin{cases} \rho\partial_{tt}\boldsymbol{u}^* - \mu\Delta\boldsymbol{u}^* - (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}^*) + \gamma T_0\nabla\partial_t\theta^* & = N_T(\boldsymbol{f}) - \boldsymbol{X}_T \quad \text{in } Q_T \\ -\rho C_s\partial_t\theta^* - \kappa\Delta\theta^* - \int_t^T k(t-s)\Delta\theta^*(s)\,\mathrm{d}s & \\ -\gamma\nabla\cdot\boldsymbol{u}^* & = 0 \quad \text{in } Q_T \\ \boldsymbol{u}^* = \boldsymbol{0},\ \theta^* & = 0 \quad \text{in } \Sigma_T \\ \boldsymbol{u}^*(\cdot, T) = \boldsymbol{0},\ \partial_t\boldsymbol{u}^*(\cdot, T) & = \boldsymbol{0} \quad \text{in } \Omega \\ \theta^*(\cdot, T) & = 0 \quad \text{in } \Omega \end{cases}$$

where $\boldsymbol{u}^*$ is solution to adjoint problem

- Steepest descent

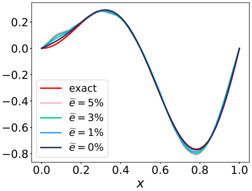
$$\begin{cases} \boldsymbol{f}_n & = \boldsymbol{f}_{n-1} - \tau_n\nabla\mathcal{I}_\beta[\boldsymbol{f}_{n-1}] \\ \tau_n & = \operatorname{argmin}_{\tau>0}\mathcal{I}_\beta(\boldsymbol{f}_{n-1} - \tau\nabla\mathcal{I}_\beta[\boldsymbol{f}_{n-1}]) \end{cases}$$

(Max. number of iterations: 200)

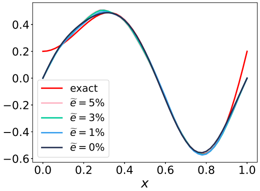
✓ fast (CGM: faster)

✓ forward and adjoint solver

✗ zero update at the boundary, β



$f(x) = x\sin(2\pi x)$, $\beta = 0$



$f(x) = x\sin(2\pi x) + 0.2$, $\beta = 0$

F. Maes, K. Van Bockstal, Numerical algorithms for the reconstruction of space-dependent sources in thermoelasticity. *Math.*

Reconstruction for inverse source problems in thermoelasticity

- Sobolev gradient method

  - Update with $\nabla_S \mathcal{I}_\beta \in \mathbf{H}^1(\Omega)$ as (weak) solution to

$$\begin{cases} -\nabla \cdot (r_1 \nabla \mathcal{K}) + r_0 \mathcal{K} = \nabla \mathcal{I}_\beta & \text{in } \Omega \\ \nabla \mathcal{K} \boldsymbol{n} = \mathbf{0} & \text{on } \partial\Omega \end{cases}$$

$$\begin{cases} \boldsymbol{f}_n = \boldsymbol{f}_{n-1} - \tau_n \nabla_S \mathcal{I}_\beta[\boldsymbol{f}_{n-1}] \\ \tau_n = \operatorname{argmin}_{\tau>0} \mathcal{I}_\beta(\boldsymbol{f}_{n-1} - \tau \nabla_S \mathcal{I}_\beta[\boldsymbol{f}_{n-1}]) \end{cases}$$

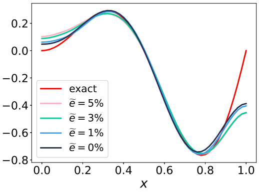
✓ fast (CGM: faster)

✓ forward, adjoint and elliptic solver

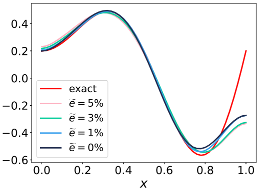
✓ updates at the boundary

✗ Choice of r_1, r_0, β

$f(x) = x\sin(2\pi x)$,
$\beta = 0, r_0 = 1, r_1 = 1/100$



$f(x) = x\sin(2\pi x) + 0.2$,
$\beta = 0, r_0 = 1, r_1 = 1/100$



F. Maes, K. Van Bockstal Numerical algorithms for the reconstruction of space-dependent sources in thermoelasticity. *Math. Comput. Simul.* ***236***, *426–454* (2025)

Future research directions

- **Open problem:** Reconstruction of a spatially varying heat source from final-time temperature data
- Further development and analysis of the **Sobolev gradient method**
- Inverse problems with **incomplete boundary data** for dynamic thermoelastic systems

Thank you for your attention!

Questions and comments are very welcome.

References I

Cannon, J. (1968).
Determination of an unknown heat source from overspecified boundary data.
SIAM Journal on Numerical Analysis, 5(2):275–286.

Erdem, A., Lesnic, D., and Hasanov, A. (2013).
Identification of a spacewise dependent heat source.
Applied Mathematical Modelling, 37(24):10231–10244.

Isakov, V. (1990).
Inverse source problems.
Providence, RI: American Mathematical Society.

Johansson, B. T. and Lesnic, D. (2007).
A variational method for identifying a spacewise-dependent heat source.
IMA Journal of Applied Mathematics, 72(6):748–760.

Logg, A., Mardal, K.-A., Wells, G. N., et al. (2012a).
Automated Solution of Differential Equations by the Finite Element Method.
Springer, Berlin, Heidelberg.

Logg, A. and Wells, G. N. (2010).
DOLFIN: Automated Finite Element Computing.
ACM Trans. Math. Software, 37(2):28.

References II

Logg, A., Wells, G. N., and Hake, J. (2012b).
DOLFIN: a C++/Python Finite Element Library, chapter 10.
Springer, Berlin, Heidelberg.

Prilepko, A. I. and Solov'ev, V. V. (1987).
Solvability theorems and Rothe's method for inverse problems for a parabolic equation. I.
Differ. Equations, 23(10):1230–1237.

Rundell, W. (1980).
Determination of an unknown non-homogeneous term in a linear partial differential equation from overspecified boundary data.
Applicable Analysis, 10(3):231–242.

Slodička, M. (2020).
Uniqueness for an inverse source problem of determining a space dependent source in a non-autonomous parabolic equation.
Applied Mathematics Letters, 107:106395.

Slodička, M. and Johansson, B. T. (2016).
Uniqueness and counterexamples in some inverse source problems.
Applied Mathematics Letters, 58:56–61.

Solov'ev, V. V. (1989).
Solvability of the inverse problem of finding a source, using overdetermination on the upper base for a parabolic equation.
Differ. Equations, 25(9):1114–1119.