Adaptive iterative approximation in nonlinear PDEs II

Martin Vohralík

in collaboration with

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Outline

1. Introduction
   - Nonsmooth and degenerate nonlinearities
   - Variational inequalities (complementarity problems)
   - Setting & achievements
2. Heat equation: robustness wrt final time and space–time error localization
   - A posteriori error estimates
   - Flux reconstruction
3. The Richards equation: adaptive regularization and linearization
   - Discretization, regularization, linearization
   - Flux reconstruction
   - A posteriori estimates and adaptive iterative approximation
4. Multiphase multicompositional flows (phase appearance and disappearance)
   - Adaptive iterative approximation
5. Conclusions

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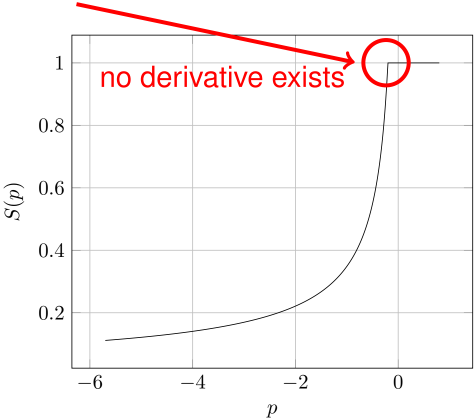
Nonsmooth and degenerate nonlinearities

Nonsmooth **nonlinearities**



Brooks–Corey pressure–saturation function

Nonsmooth and degenerate nonlinearities

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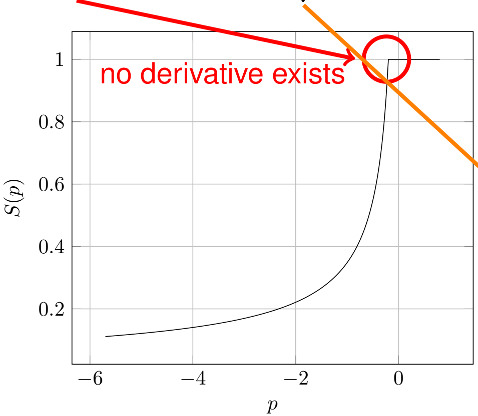

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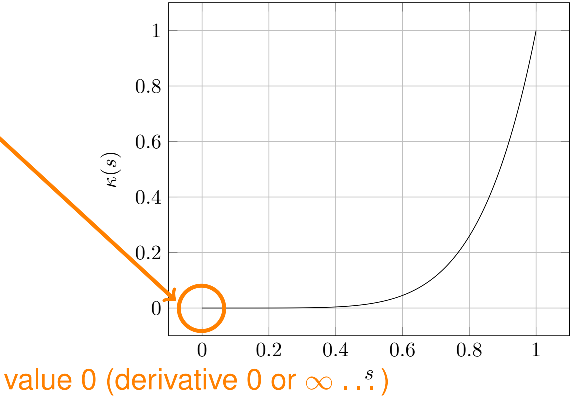

Brooks–Corey saturation–relative permeability function

Nonsmooth and degenerate nonlinearities

Nonsmooth and degenerate nonlinearities

- omnipresent in applications
- cause **convergence troubles** of **standard iterative linearization methods**

Nonsmooth and degenerate nonlinearities: common recipes

Nonsmooth and degenerate nonlinearities

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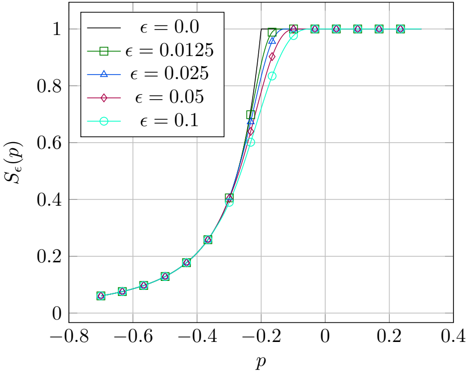
- cause **convergence troubles** of **standard iterative linearization methods**

Common recipes

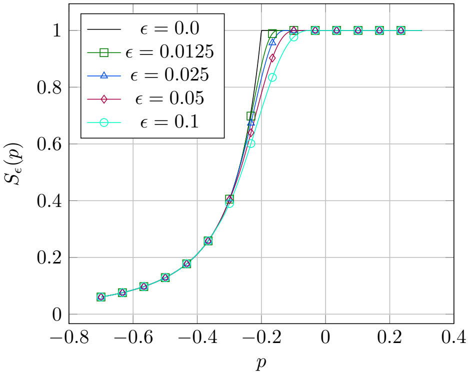
- timestep cutting
- damping
- scheme switching (from Newton to fixed-point . . .)
- variable switching
- regularization
- . . .

Regularization



Brooks–Corey **regularized** pressure–saturation functions

Regularization



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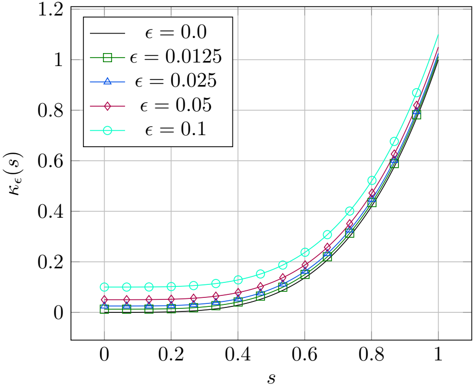

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Variational inequalities (complementarity problems)

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$$\boldsymbol{F}(\boldsymbol{X}) = \mathbf{0},$$
$$\boldsymbol{K}(\boldsymbol{X}) \geq \mathbf{0},\ \boldsymbol{G}(\boldsymbol{X}) \geq \mathbf{0},\ \boldsymbol{K}(\boldsymbol{X}) \cdot \boldsymbol{G}(\boldsymbol{X}) = 0$$

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Common methods

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- path finding
- interior-point methods
- regularization
- ...

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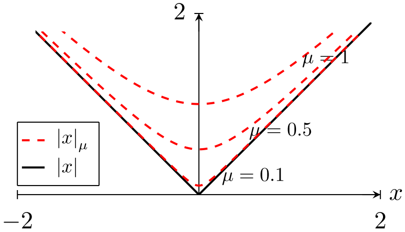
Complementarity functions

$$\boldsymbol{F}(\boldsymbol{X}) = \boldsymbol{0},$$
$$\boldsymbol{C}(\boldsymbol{X}) = \boldsymbol{0}$$

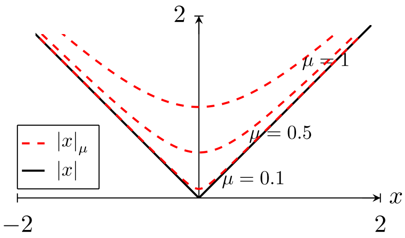
equivalent reformulation as nonlinear **nonsmooth equalities**

Regularized complementarity functions



Regularized absolute value
(Newton-min) *C*-functions

Regularized complementarity functions



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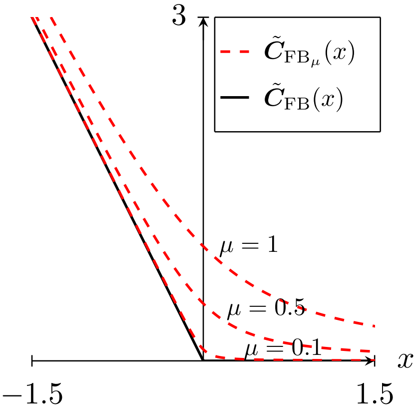

Regularized Fischer–Burmeister *C*-functions

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Numerical approximations of unsteady nonlinear PDEs:

Setting

- u: unknown exact PDE solution
- u_ℓ^n : known numerical approximation on space mesh $\mathcal{T}_\ell$ and time t^n

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Numerical approximations of unsteady nonlinear PDEs: **3 crucial questions**

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1. How **large** is the overall **error** between u and $u_\ell^{n,j,k,i}$?

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Suggested answers

1. Computable **a posteriori** error **estimates**.

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Suggested answers

1. Computable **a posteriori** error **estimates**.
2. Identification of **error components**.
3. **Balancing** error components, **adaptivity** (working where needed).

Main achievements

A posteriori error estimates

Guaranteed a posteriori error estimates

$$\sum_{n=1}^{N} \int_{t^{n-1}}^{t^n} |||u - u_\ell^{n,j,k,i}|||^2 \leq \sum_{n=1}^{N} \sum_{K \in \mathcal{T}_\ell^n} \eta_K(u_\ell^{n,j,k,i})^2$$

Main achievements

A posteriori error estimates

Guaranteed a posteriori error estimates **efficient**

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A posteriori error estimates

Guaranteed a posteriori error estimates **efficient**, **robust** with respect to the **final time**

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C_{eff} independent of T,

Main achievements

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Guaranteed a posteriori error estimates **locally space-time efficient**, **robust** with respect to the **final time**

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Holistic approach: interplay PDE–numerics–linearization–algebra

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Nonsmooth and degenerate nonlinearities/variational inequalities

Algorithm (adaptive iterative regularization)

1. regularization parameter $\epsilon_j > 0$
2. replace the nonsmooth and degenerate functions by smooth and nondegenerate ϵ_j-approximations
3. a few steps of Newton linearization (gentle nonlinearity, good initial guess)
4. decrease ϵ_j and go back to step 2

Steering

- **a posteriori estimates** of **error components**
- **algebraic** error is below linearization: stop algebraic solver
- **linearization** error is below regularization: stop Newton iterations
- **regularization** error is below discretization: stop regularization (ϵ_j is **never brought to zero**)
- **space mesh & time step adaptivity**
- **discretization** is below a specified tolerance: finish

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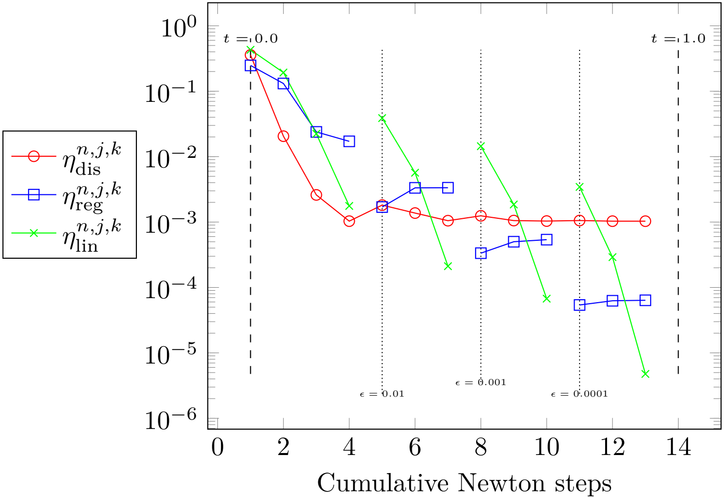
Example overall behavior (Richards equation, 1 time step)



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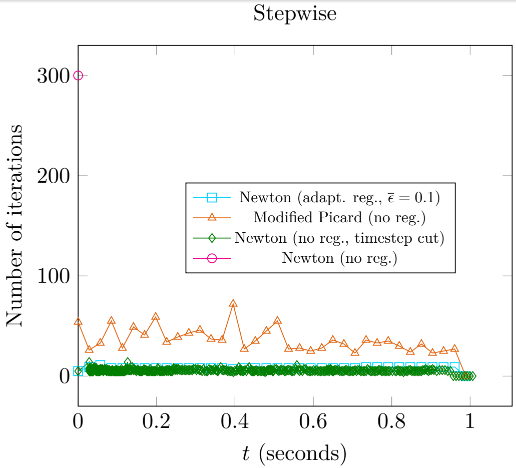
Comparison with existing approaches



Number of linearization iterations on each time step

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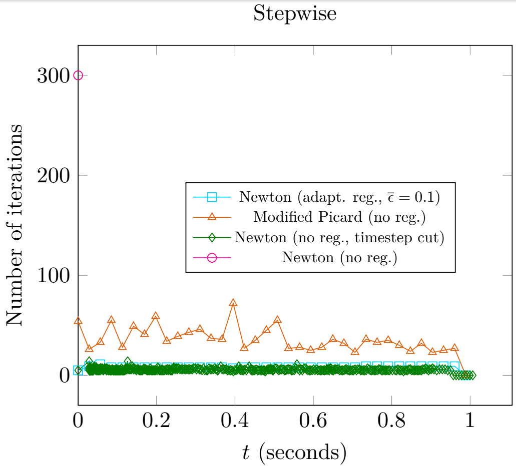

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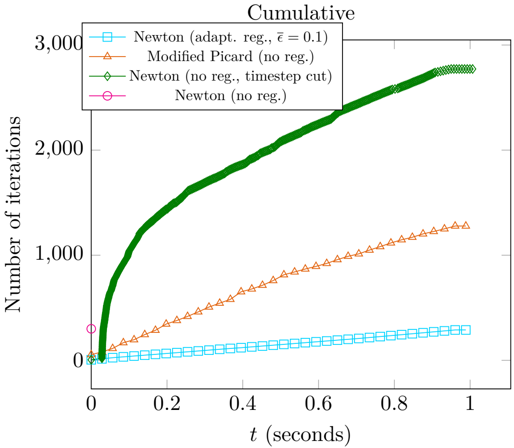

Cumulative number of linearization iterations

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The heat equation ($f \in L^2(0,T;L^2(\Omega))$, $u_0 \in L^2(\Omega)$)

The heat equation

$$\begin{aligned} \partial_t u - \Delta u &= f \quad \text{in } \Omega \times (0,T), \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) &= u_0 \quad \text{in } \Omega \end{aligned}$$

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Definition (Weak solution)

$u \in Y$ with $u(0) = u_0$ such that

$$\int_0^T \langle \partial_t u, v\rangle + (\nabla u, \nabla v)\,\mathrm{d}t = \int_0^T (f,v)\,\mathrm{d}t \qquad \forall v \in X.$$

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Nonsymmetry

Trial space Y, test space X.

The heat equation ($f \in L^2(0,T;L^2(\Omega))$, $u_0 \in L^2(\Omega)$)

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Spaces

$$X := L^2(0,T;H_0^1(\Omega)), \|v\|_X^2 := \int_0^T \|\nabla v\|^2 \,\mathrm{d}t,$$

$$Y := L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \|v\|_Y^2 := \int_0^T \|\partial_t v\|_{H^{-1}(\Omega)}^2 + \|\nabla v\|^2 \,\mathrm{d}t + \|v(T)\|^2$$

Y norm error is the dual X norm of the residual + initial condition error

$$\|u - u_{h\tau}\|_Y^2 = \sup_{v \in X,\, \|v\|_X = 1} \left[\int_0^T (f,v) - \langle \partial_t u_{h\tau}, v \rangle - (\nabla u_{h\tau}, \nabla v) \,\mathrm{d}t \right]^2 + \|u_0 - u_{h\tau}(0)\|^2$$

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- Setting & achievements

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- Flux reconstruction

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- Adaptive iterative approximation

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The heat equation ($f \in L^2(0,T;L^2(\Omega))$, $u_0 \in L^2(\Omega)$)

The heat equation

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\begin{aligned}
\partial_t u - \Delta u &= f \quad \text{in } \Omega \times (0,T),\\
u &= 0 \quad \text{on } \partial\Omega \times (0,T),\\
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\end{aligned}
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- C_{eff} a generic constant independent of Ω, u, $u_{h\tau}$, ℓ, p, τ_n, q,

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Guaranteed error upper bound (reliability) ($u_{h\tau}$ FE in space, DG in time approx.)

$$\underbrace{|||u - u_{h\tau}|||}_{\text{unknown error}} \leq \underbrace{\eta(u_{h\tau})}_{\text{computable estimator}}$$

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Robust local in space and in time error lower bound (efficiency)

$$\eta_{K,I_n}(u_{h\tau}) \leq C_{\mathrm{eff}} |||u - u_{h\tau}|||_{\omega_K \times I_n}$$

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Equilibrated flux reconstruction

Definition (Equilibrated flux reconstruction)

For each time-step interval I_n and for each vertex $\mathbf{a} \in \mathcal{V}^n$, let

$$\boldsymbol{\sigma}_{h\tau}^{\mathbf{a},n} := \arg \min_{\substack{\mathbf{v}_\ell \in \mathbf{V}_{h\tau}^{\mathbf{a},n} \\ \nabla\cdot\mathbf{v}_\ell = \psi_{\mathbf{a}}(f - \partial_t \mathcal{I} u_{h\tau}) - \nabla\psi_{\mathbf{a}}\cdot\nabla u_{h\tau}}} \int_{I_n} \|\mathbf{v}_\ell + \psi_{\mathbf{a}} \nabla u_{h\tau}\|_{\omega_{\mathbf{a}}}^2 \, \mathrm{d}t.$$

Then set

$$\boldsymbol{\sigma}_{h\tau} := \sum_{n=1}^{N} \sum_{\mathbf{a}\in\mathcal{V}^n} \boldsymbol{\sigma}_{h\tau}^{\mathbf{a},n}.$$

Comments

- satisfies $\boldsymbol{\sigma}_{h\tau} \in L^2(0,T;\mathbf{H}(\mathrm{div},\Omega))$ with $\nabla\cdot\boldsymbol{\sigma}_{h\tau} = f - \partial_t \mathcal{I} u_{h\tau}$
- a priori a local space-time problem, $\mathbf{V}_{\ell\tau}^{\mathbf{a},n} := \mathcal{Q}_q(I_n; \mathbf{V}_\ell^{\mathbf{a},n})$
- uncouples to q elliptic problems posed in $\mathbf{V}_\ell^{\mathbf{a},n}$

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Modelling flow of water and air through soil

The Richards equation

Find $p : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t S(p) - \nabla\cdot[\boldsymbol{K}\kappa(S(p))(\nabla p + \boldsymbol{g})] &= f && \text{in } \Omega \times (0,T), \\ p &= 0 && \text{on } \partial\Omega \times (0,T), \\ (S(p))(\cdot,0) &= s_0 && \text{in } \Omega. \end{aligned}$$

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Setting

- p: pressure
- $S(p)$: saturation
- $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, open polytope with Lipschitz boundary $\partial\Omega$
- T: final time
- diffusion tensor $\mathbf{K}$, source term $f \in C^1([0, 1])$, gravity $\mathbf{g}$, initial saturation $s_0 \in L^\infty(\Omega)$, $0 \le s_0 \le 1$
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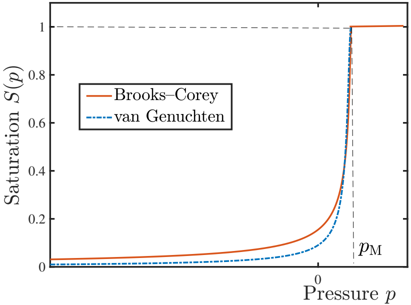

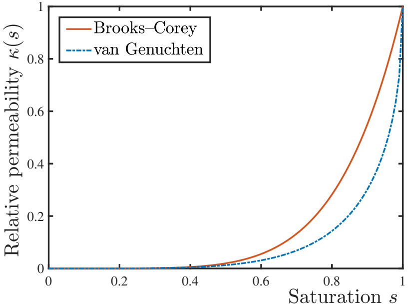

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Backward Euler & finite element discretization

Piecewise polynomial space

$$V_\ell^0 := \mathcal{P}_1(\mathcal{T}_\ell) \cap H_0^1(\Omega)$$

Discretization: finite elements & backward Euler

For each $1 \le n \le N$, given $p_\ell^{n-1} \in V_\ell^0$, find the approximate pressure $p_\ell^n \in V_\ell^0$ satisfying

$$\frac{1}{\tau^n}(S(p_\ell^n) - S(p_\ell^{n-1}), v_\ell) + (\mathbf{F}(p_\ell^n), \nabla v_\ell) = (f(\cdot, t_n), v_\ell) \qquad \forall v_\ell \in V_\ell^0,$$

where the flux is given by

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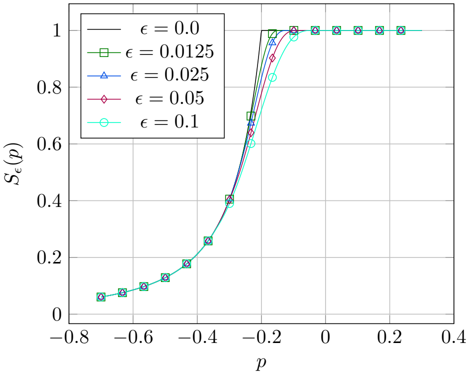

Brooks–Corey **regularized** pressure–saturation functions

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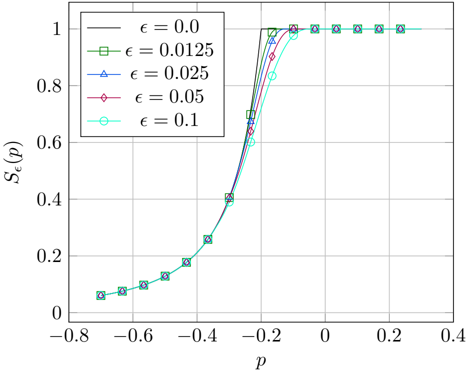

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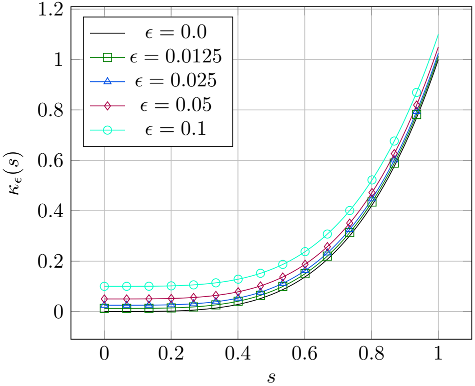

Brooks–Corey **regularized** saturation–relative permeability functions

Regularization

Regularization

Given $p_\ell^{n-1,\bar{j}} \in V_\ell^0$, find $p_\ell^{n,j} \in V_\ell^0$ satisfying

$$\frac{1}{\tau^n}(S_{\epsilon^j}(p_\ell^{n,j}) - S_{\epsilon^j}(p_\ell^{n-1,\bar{j}}), v_\ell) + (\boldsymbol{F}_{\epsilon^j}(p_\ell^{n,j}), \nabla v_\ell) = (f(\cdot, t_n), v_\ell) \quad \forall v_\ell \in V_\ell^0,$$

where the **regularized flux** is given by

$$\boldsymbol{F}_{\epsilon^j}(q) := \boldsymbol{K}\kappa_{\epsilon^j}(S_{\epsilon^j}(q))[\nabla q + \boldsymbol{g}].$$

- ϵ^j: sequence of regularization parameters
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Given an initial guess $p_\ell^{n,j,k-1}$, find $p_\ell^{n,j,k} \in V_\ell^0$ such that, for all $v_\ell \in V_\ell^0$,

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- $\bar{k}$: stopping linearization index
- modified Picard:

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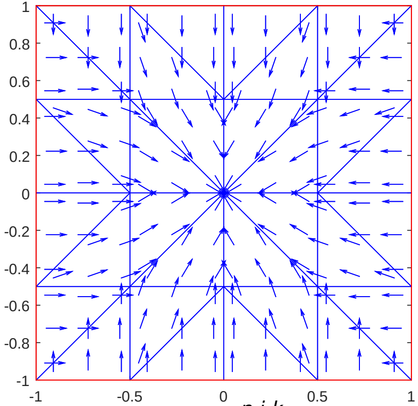
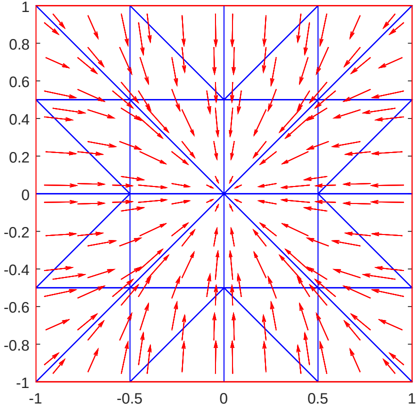
Flux reconstruction: $-\boldsymbol{F}_\ell^{n,j,k} \notin \boldsymbol{H}(\mathrm{div},\Omega) \to \boldsymbol{\sigma}_\ell^{n,j,k} \in \mathcal{RT}_0(\mathcal{T}_\ell) \cap \boldsymbol{H}(\mathrm{div},\Omega)$



Flux $-\boldsymbol{F}_\ell^{n,j,k}$



Equilibrated flux reconstruction $\boldsymbol{\sigma}_\ell$

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A posteriori estimates of error components

A posteriori estimates of error components

$$\eta_{\mathrm{dis}}^{n,j,k} := \| \mathbf{F}_\ell^{n,j,k} + \boldsymbol{\sigma}_\ell^{n,j,k} \| \qquad \text{(discretization)}$$

$$\eta_{\mathrm{lin}}^{j,k} := \| \mathbf{F}_{\epsilon^j}(p_\ell^{n,j,k}) - \mathbf{F}_\ell^{n,j,k} \| \qquad \text{(linearization)}$$

$$\eta_{\mathrm{reg}}^{j,k} := \| \mathbf{F}(p_\ell^{n,j,k}) - \mathbf{F}_{\epsilon^j}(p_\ell^{n,j,k}) \| \qquad \text{(regularization)}$$

Adaptive regularization and linearization

Adaptive regularization and linearization ($\gamma_{\rm lin}, \gamma_{\rm reg} \approx 0.1$)

$$\eta_{\rm lin}^{n,j,\bar{k}} < \gamma_{\rm lin} \eta_{\rm reg}^{n,j,\bar{k}}$$

$$\eta_{\rm reg}^{n,\bar{j},\bar{k}} < \gamma_{\rm reg} \eta_{\rm dis}^{n,\bar{j},\bar{k}}$$

Strictly unsaturated medium

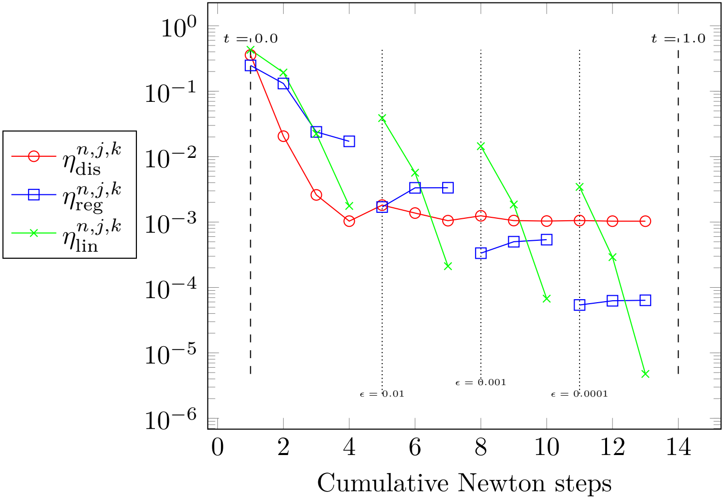
- $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 = (0,1) \times (0,1/4]$, $\Omega_2 = (0,1) \times (1/4,1)$
- $T = 1$, $\boldsymbol{K} = \boldsymbol{I}$, $\boldsymbol{g} = (0,1)^T$
- effective saturation $\mathscr{S}(s) = \frac{s - S_{\mathrm{R}}}{S_{\mathrm{V}} - S_{\mathrm{R}}}$
- van Genuchten model

$$\kappa(s) = \kappa_{\mathrm{c}} \sqrt{\mathscr{S}(s)}(1 - (1 - \mathscr{S}(s)^{1/\lambda_2})^{\lambda_2})^2,$$

$$S(p) = \begin{cases} \left[(1 + (-\alpha p)^{\frac{1}{1-\lambda_2}} \right]^{-\lambda_2} & p \le p_{\mathrm{M}}, \\ 1 & p > p_{\mathrm{M}} \end{cases}$$

- $p_{\mathrm{M}} = 0$, $S_{\mathrm{R}} = 0.026$, $S_{\mathrm{V}} = 0.42$, $\kappa_{\mathrm{c}} = 0.12$, $\alpha = 0.551$, $\lambda_2 = 0.655$
- $f(x,y) = \begin{cases} 0 & (x,y) \in \Omega_1, \\ 0.06 \cos(\frac{4}{3}\pi y) \sin(x) & (x,y) \in \Omega_2 \end{cases}$
- $p_0(x,y) = \begin{cases} -y - 1/4 & (x,y) \in \Omega_1, \\ -4 & (x,y) \in \Omega_2 \end{cases}$
- $s_0 = S(p_0)$
- uniform mesh with $40 \times 40 \times 2$ elements, $\tau^0 = 1$

Adaptive regularization and linearization



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Injection case

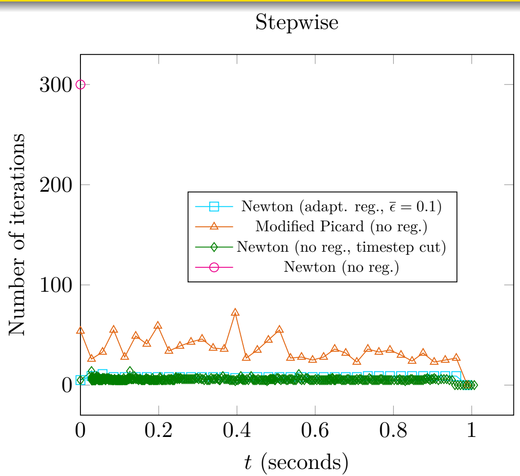
- $\Omega = (0,1)^2$
- $T = 1, \boldsymbol{K} = \boldsymbol{I}, \boldsymbol{g} = (0,-1)^T$
- effective saturation $\mathscr{S}(s) = \frac{s - S_{\mathrm{R}}}{S_{\mathrm{V}} - S_{\mathrm{R}}}$
- Brooks–Corey model

$$\kappa(s) = \mathscr{S}(s)^{\frac{2+3\lambda_1}{\lambda_1}},$$

$$S(p) = \begin{cases} (-p/p_{\mathrm{M}})^{-\lambda_1} & p \leq p_{\mathrm{M}}, \\ 1 & p > p_{\mathrm{M}} \end{cases}$$

- $p_{\mathrm{M}} = -0.2$, $\lambda_1 = 2.239$
- $f = 0$
- $p_0 = -1$
- $s_0 = S(p_0)$
- quasi uniform mesh with $\ell = 2.82 \cdot 10^{-2}$, $\tau^0 = 2.82 \cdot 10^{-2}$

Do we reduce the **computational cost**?



Number of linearization iterations on each time step

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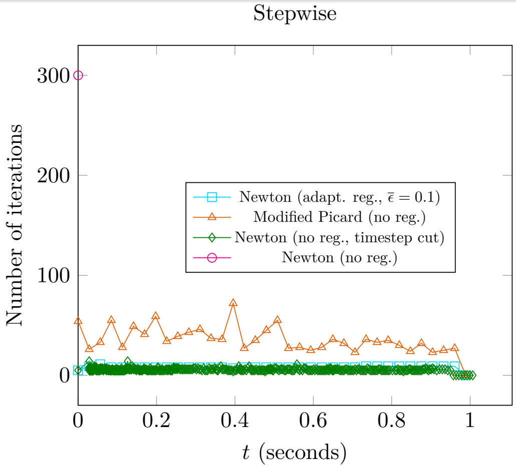

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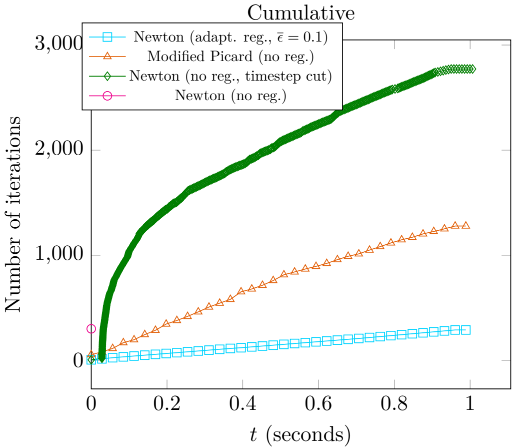

Cumulative number of linearization iterations

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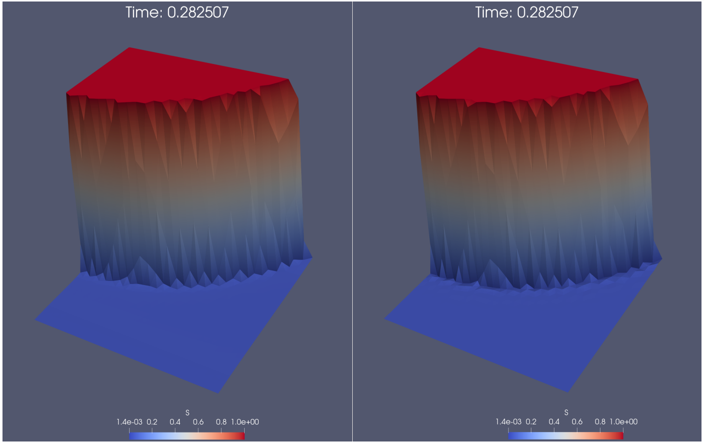
Do we **lose precision**?



Saturation field $s = S(p_\ell^{n,\bar{j},\bar{k}})$ using Newton's method and adaptive regularization $\epsilon^1 = 0.1$ (left) and modified Picard with no regularization (right)

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Realistic case

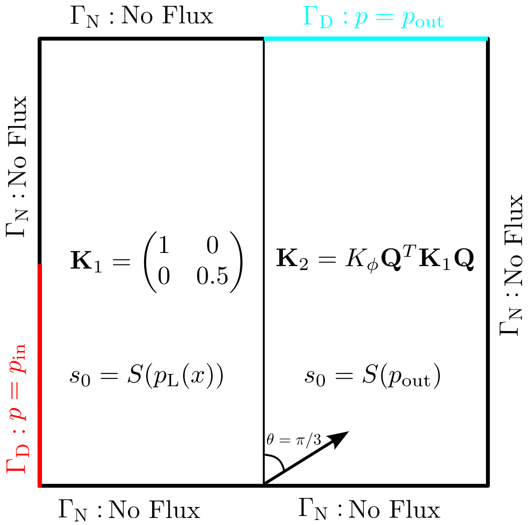
- $\Omega = (0,1)^2$
- $T = 1$
- $\boldsymbol{g} = (-1, 0)^T$
- $\mathbf{Q} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
- $K_\phi = 0.1$
- effective saturation $\mathscr{S}(s) = \frac{s - S_\mathrm{R}}{S_\mathrm{V} - S_\mathrm{R}}$
- Brooks–Corey model

$$\kappa(s) = \mathscr{S}(s)^{\frac{2+3\lambda_1}{\lambda_1}},$$

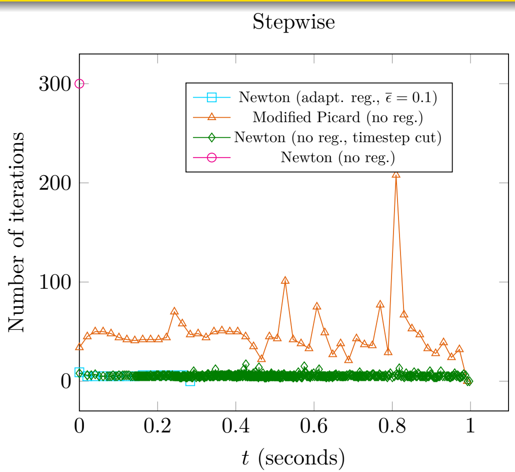
$$S(p) = \begin{cases} (-p/p_\mathrm{M})^{-\lambda_1} & p \le p_\mathrm{M}, \\ 1 & p > p_\mathrm{M} \end{cases}$$

- $p_\mathrm{M} = -0.2$, $\lambda_1 = 2$
- $f = 0$
- quasi uniform mesh with $h = 2.02 \cdot 10^{-2}$, $\tau^0 = 2.02 \cdot 10^{-2}$
- $p_\mathrm{L}(\boldsymbol{x}) = \left(\frac{p_\mathrm{out} - p_\mathrm{in}}{0.5}\right) \boldsymbol{x}$, $p_\mathrm{out} = -2.0$, $p_\mathrm{in} = -0.2$, $p_\mathrm{D} = p_0|_{\Gamma_\mathrm{D}}$

Realistic case setting



Do we reduce the **computational cost**?



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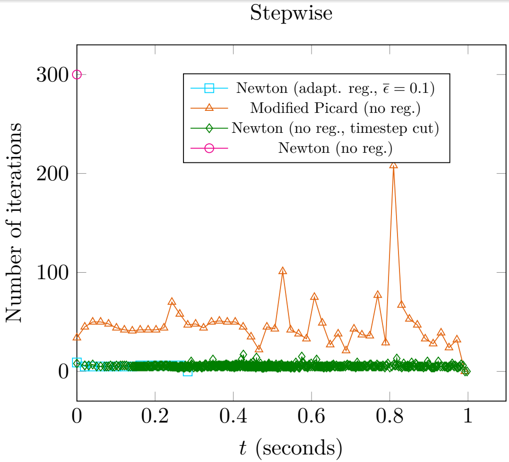

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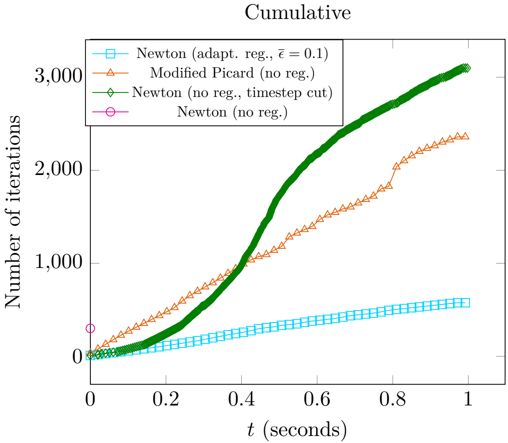

Cumulative number of linearization iterations

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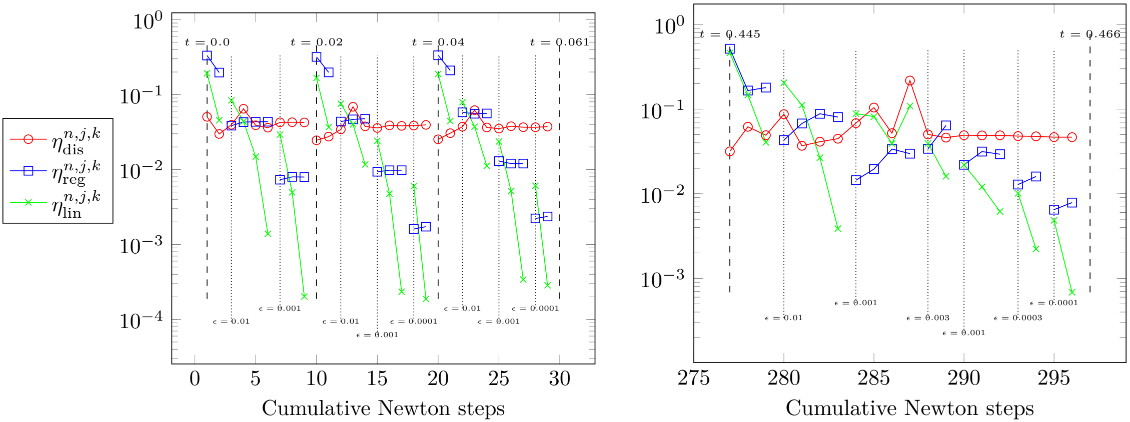
Adaptive regularization and linearization



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Perched water table case

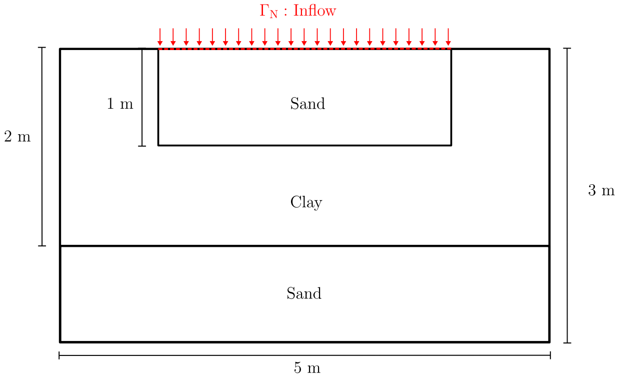
- $\Omega = (-2.5 \text{ m}, 2.5 \text{ m}) \times (-3 \text{ m}, 0 \text{ m})$
- $T = 86400$ s (one day)
- $\mathbf{K} = \boldsymbol{I}$
- $\boldsymbol{g} = (-1, 0)^T$
- effective saturation $\mathscr{S}(s) = \frac{s - S_{\rm R}}{S_{\rm V} - S_{\rm R}}$
- van Genuchten model

$$\kappa(s) = \kappa_{\rm c} \sqrt{\mathscr{S}(s)} (1 - (1 - \mathscr{S}(s)^{1/\lambda_2})^{\lambda_2})^2,$$

$$S(p) = \begin{cases} \left[(1 + (-\alpha p)^{\frac{1}{1-\lambda_2}} \right]^{-\lambda_2} & p \leq p_{\rm M}, \\ 1 & p > p_{\rm M} \end{cases}$$

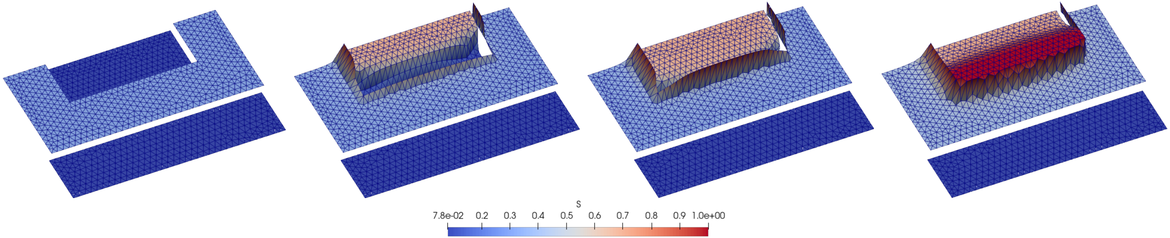
- $f = 0$
- quasi uniform mesh with $h = 8.2 \cdot 10^{-2}$
- $\tau^0 = 60$ s, (increase $\tau^n := 1.2\tau^{n-1}$ for $n \geq 1$)
- initial condition $s_0 = S(p_0)$ with $p_0 = -300$ m

Perched water table case setting



Material	κ_c	ϕ	S_R	S_V	λ_2	α
Sand	6.262×10^{-5}	0.368	0.07818	1	0.553	2.8
Clay	1.516×10^{-6}	0.4686	0.2262	1	0.2835	1.04

Perched water table case saturation evolution



Saturation at $t = 0$ s, $21 \cdot 10^3$ s, $41 \cdot 10^3$ s, $86.1 \cdot 10^3$ s

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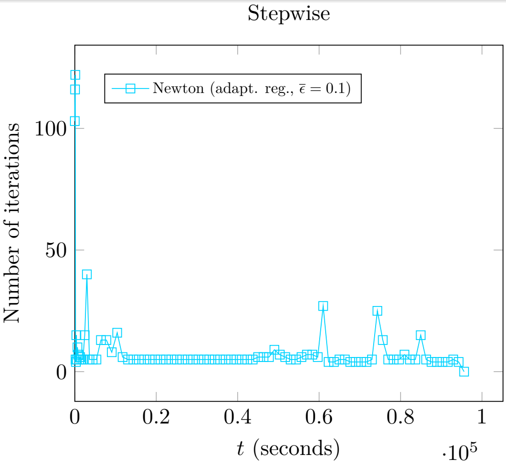
Performance: only adaptive regularization and linearization works



Number of linearization iterations on each time step

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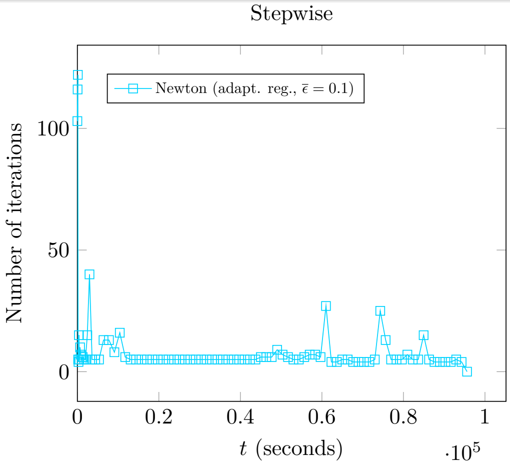

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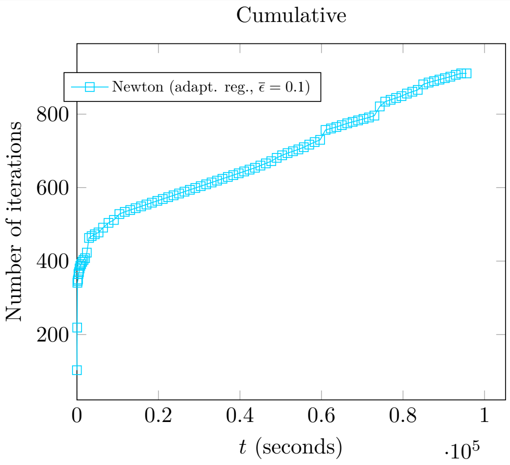

Cumulative number of linearization iterations

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Two-phase flow

Incompressible two-phase flow in porous media

Find *saturations* s_α and *pressures* p_α, $\alpha \in \{\mathrm{g}, \mathrm{w}\}$, such that

$$\begin{aligned} \partial_t(\phi s_\alpha) - \nabla\cdot\left(\frac{k_{\mathrm{r},\alpha}(s_\mathrm{w})}{\mu_\alpha}\boldsymbol{K}(\nabla p_\alpha + \rho_\alpha g \nabla z)\right) &= q_\alpha, \qquad \alpha \in \{\mathrm{g}, \mathrm{w}\}, \\ s_\mathrm{g} + s_\mathrm{w} &= 1, \\ p_\mathrm{g} - p_\mathrm{w} &= p_\mathrm{c}(s_\mathrm{w}) \end{aligned}$$

- unsteady, nonlinear, and degenerate problem
- coupled system of PDEs & algebraic constraints

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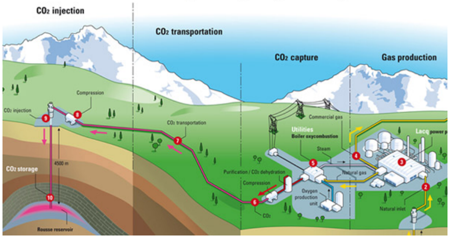
Carbon capture & geological storage



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Multi-phase multi-compositional flow

Theorem (Multi-phase multi-compositional Darcy flow with phase (dis)appearance)

There holds

$$\textit{error on time interval } I_n \leq \left\{ \sum_{c \in \mathcal{C}} \left(\eta_{\mathrm{sp},c}^{n,j,k,i} + \eta_{\mathrm{tm},c}^{n,j,k,i} + \eta_{\mathrm{reg},c}^{n,j,k,i} + \eta_{\mathrm{lin},c}^{n,j,k,i} + \eta_{\mathrm{alg},c}^{n,j,k,i} \right)^2 \right\}^{1/2}.$$

Error components

- $\eta_{\mathrm{sp},c}^{n,j,k,i}$: spatial discretization
- $\eta_{\mathrm{tm},c}^{n,j,k,i}$: temporal discretization
- $\eta_{\mathrm{reg},c}^{n,j,k,i}$: regularization
- $\eta_{\mathrm{lin},c}^{n,j,k,i}$: linearization
- $\eta_{\mathrm{alg},c}^{n,j,k,i}$: algebraic solver

Error control

- at **any moment** during the simulation
- price: sparse **matrix-vector** multiplication

Full adaptivity

- **same physical units** of all component estimators
- **balance** all component estimators
- **online steering**

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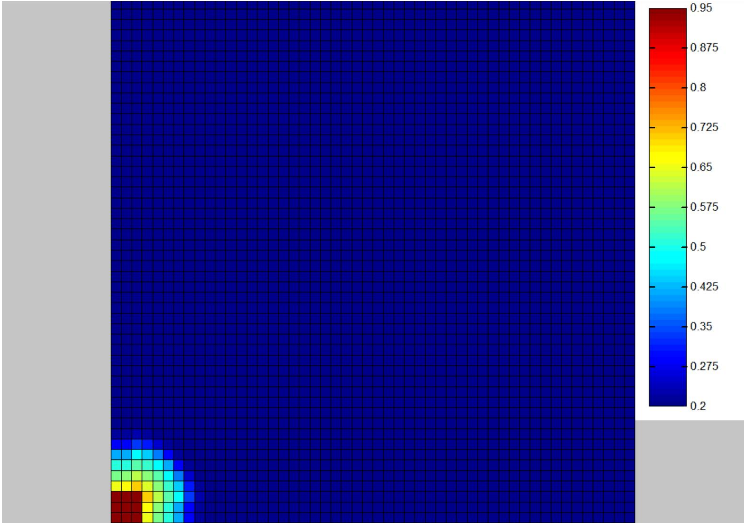
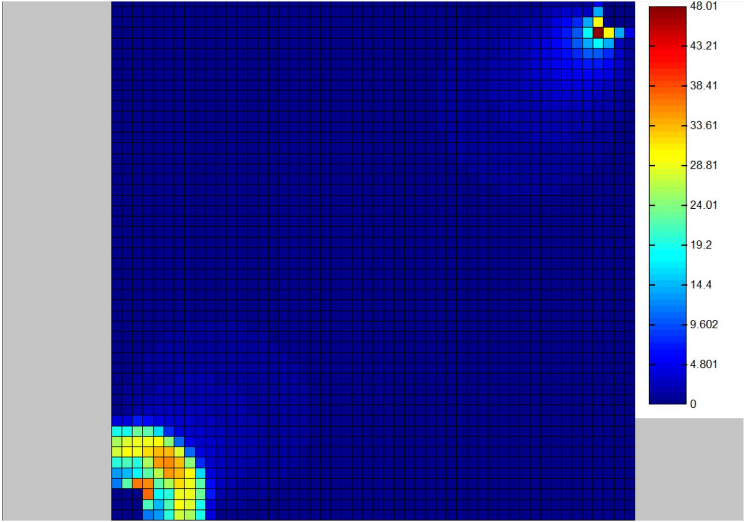
Geological sequestration of CO_2, CO_2 saturation



M. Vohralík, M. Wheeler, Computational Geosciences (2013)

Geological sequestration of CO_2, **overall a posteriori estimate**



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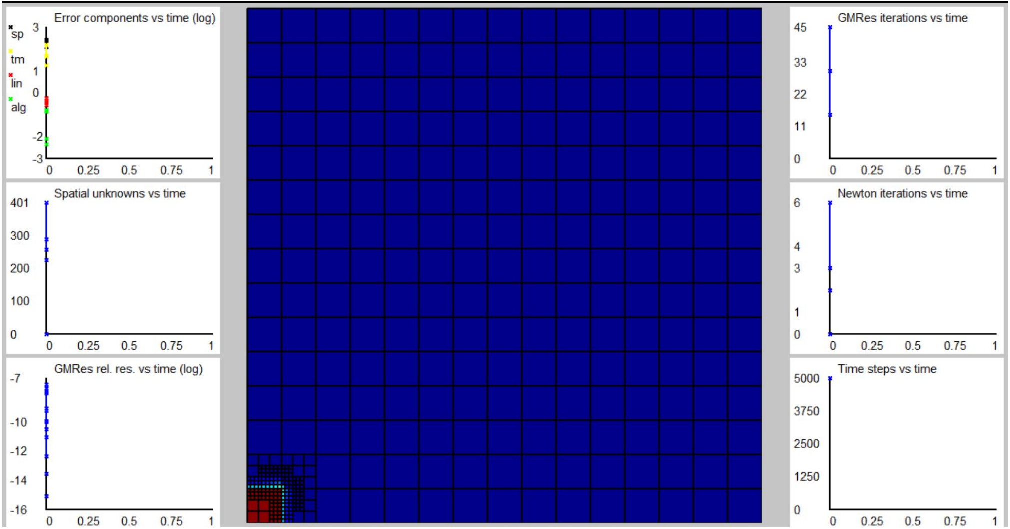
Geological sequestration of CO_2, **adaptive iterative approximation**



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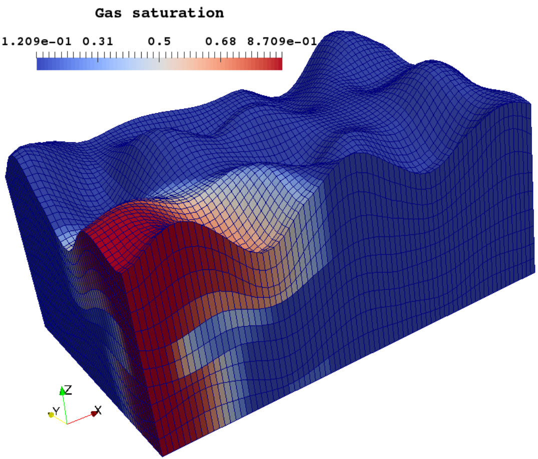
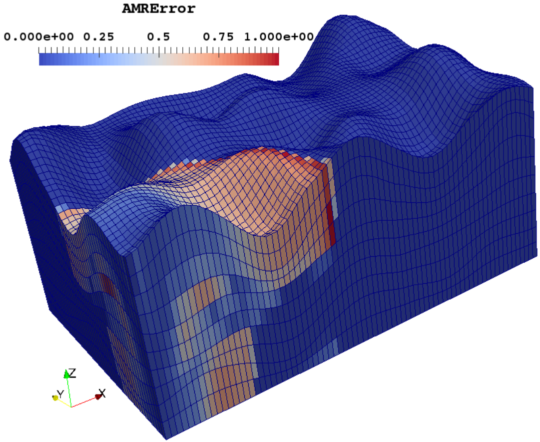
Adaptivity: 3 phases, 3 components (black-oil) problem



Gas saturation



A posteriori error estimate

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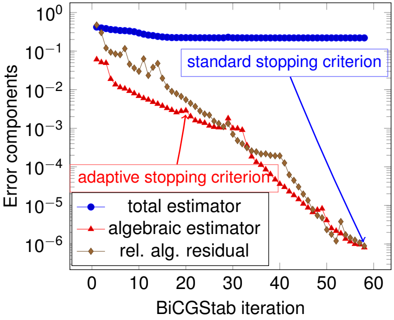
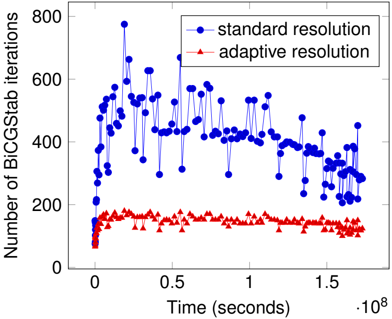
Three-phases, three-components (black-oil) problem: algebraic solver & spatial mesh adaptivity





	Linear solver steps	Resolution time	AMR time	Estimators evaluation	Gain factor
Standard resolution	66386	1023s	-	-	-
Adaptive resolution	20184	201s	42s	26s	3.8

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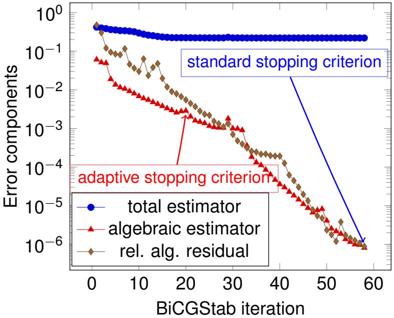

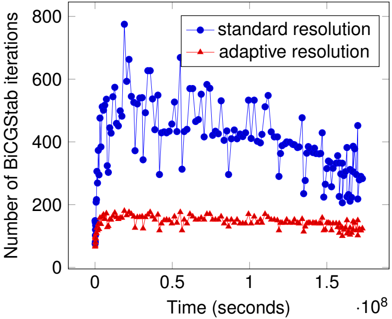

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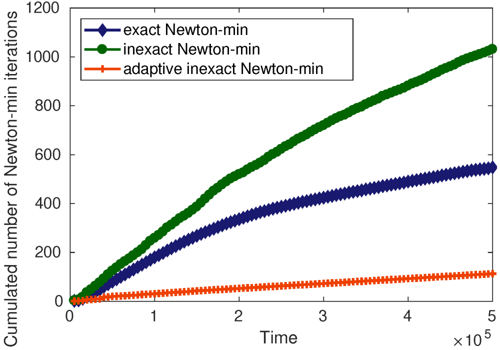
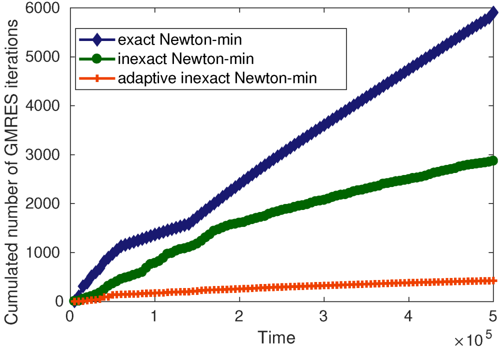
M. Vohralík, S. Yousef, Computer Methods in Applied Mechanics and Engineering (2018)

Phase (dis)appearance: Couplex-gas benchmark





Adaptive linear and nonlinear solvers

I. Ben Gharbia, J. Dabaghi, V. Martin, M. Vohralík, Computational Geosciences (2020)

Outline

1. Introduction
   - Nonsmooth and degenerate nonlinearities
   - Variational inequalities (complementarity problems)
   - Setting & achievements
2. Heat equation: robustness wrt final time and space–time error localization
   - A posteriori error estimates
   - Flux reconstruction
3. The Richards equation: adaptive regularization and linearization
   - Discretization, regularization, linearization
   - Flux reconstruction
   - A posteriori estimates and adaptive iterative approximation
4. Multiphase multicompositional flows (phase appearance and disappearance)
   - Adaptive iterative approximation
5. Conclusions

Conclusions

- a posteriori **error control**

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- **adaptive approximation**: space & time mesh,

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Thank you for your attention!