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A Discrete Exterior Calculus of Bundle-Valued Forms

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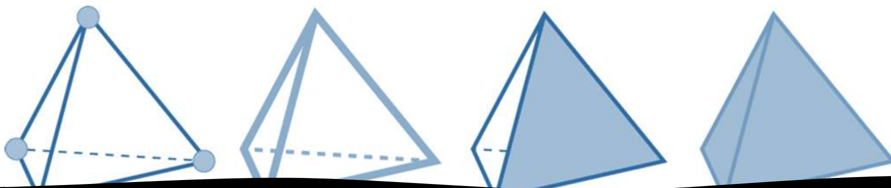
TL;DR Summary

A discrete *exterior covariant derivative operator*

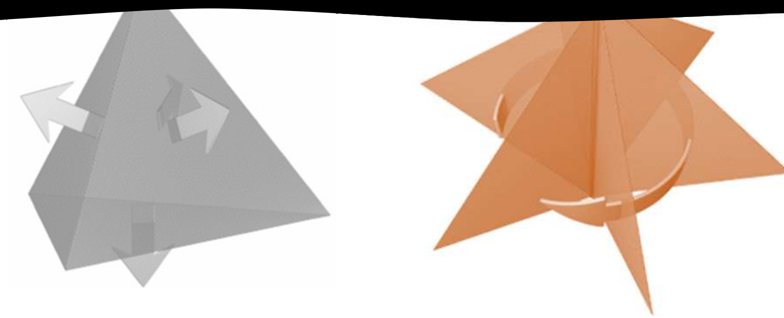
- operating on bundle-valued forms
- structure preserving (i.e., Bianchi identities are tautologies)
- extending DEC quite directly
  - [Berwick-Evans, Hirani, Schubel 2021] cracked it!
  - on second thought, too combinatorial to be perfect…

Our contributions:

- *Identifying crucial role of frame fields*
  - evaluation involves non-commutative composition of //-transport
  - discretization *must* account for local frame field choice
- *Enforcing convergence under refinement*
  - Bianchi identities exactly satisfied for any resolution is great…
  - but we need correct evaluations in the limit too
    - must understand how discrete and continuous forms are related



Preamble: Discrete Exterior Calculus



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Discrete Exterior Calculus

with Anil Hirani, then Melvin Leok , Tom Tyranowski,Ari Stern

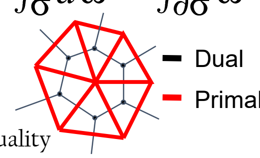
Foundations: discrete differential forms

- mesh as computational structure
  - chains as proxies for domains

  - store k-forms as integrated values over simplices
    - cochains extend point sampling to “simplex sampling”
- basic operators: d (exterior derivative) and $\star$ (Hodge star) through heavy use of **adjointness**
  - d through Stokes

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega$$

    - d is a topological operator, hence **exact**
  - exact link to (co)homology [Munkres]
  - simplest Hodge duality via **mesh duality**
    - exploits (weighted) Delaunay/Voronoi duality



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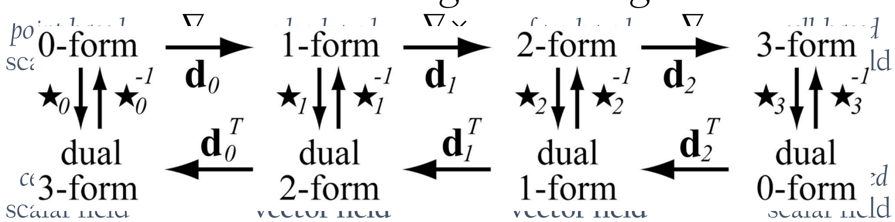
Discrete De Rham Sequence

Discrete calculus through linear algebra:



- simple exercise in matrix assembly
- discrete Hodge theory particularly simple
- Whitney basis fcts extending FE picture [Bossavit]
- can be made higher-order or spectral accurate too!
  - FEEC, subdivision EC, isogeometric analysis, etc
  - even for non-flat cell complexes, power duals, etc...

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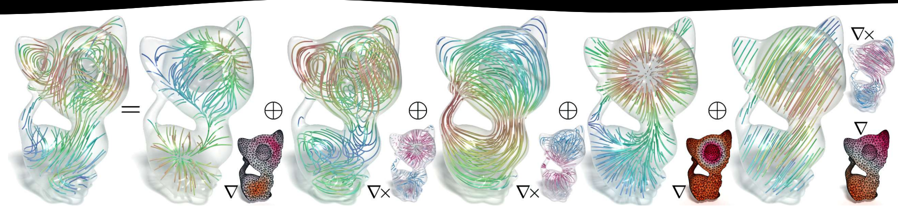
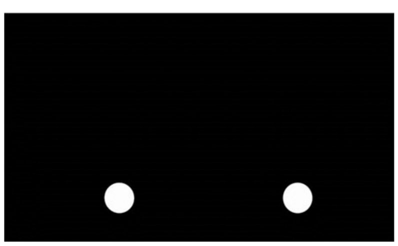
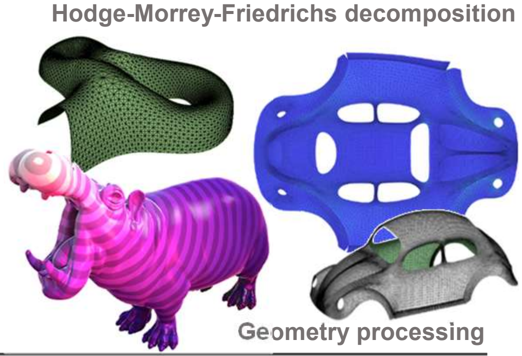
Lots of Applications



Hodge-Morrey-Friedrichs decomposition



Navier-Stokes simulation



Geometry processing

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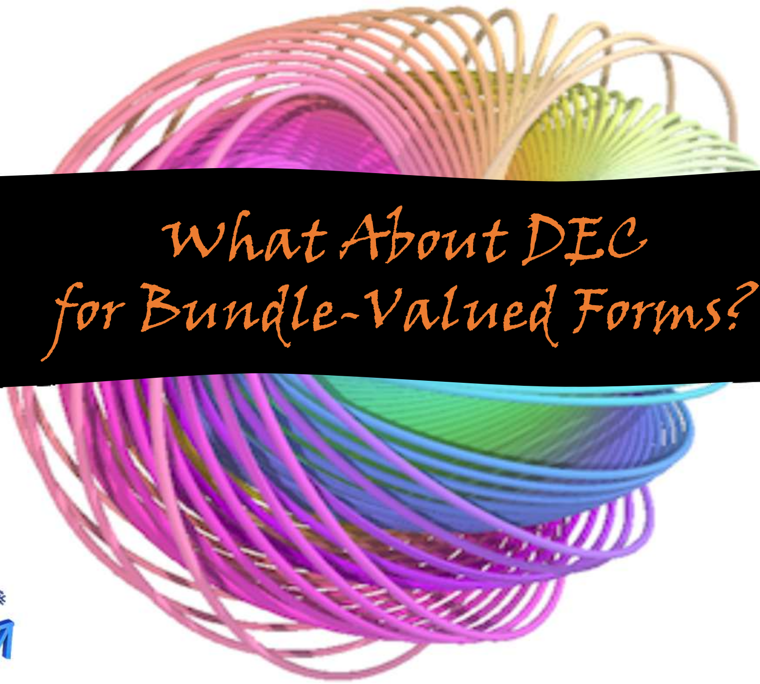

What About DEC for Bundle-Valued Forms?

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Continuous Notions At A Glance

Connection on a vector bundle $\pi : E \to M$: $\nabla = d + \omega$

- ω : local connection 1-form *(depends on frame field!* $\nabla f_a = f_b \omega_a^b$)
- parallel transport along curve between fibers: $\mathcal{R}_t : E_{\gamma(t)} \to E_{\gamma(0)}$
- π induces ∇^{End} on endomorphism bundle $\text{End}(E) \to M$

Covariant exterior derivative

$$d^\nabla \alpha = d\alpha + \omega \wedge \alpha \quad \forall \alpha \in \Omega^k(M, E)$$

- curvature 2-form: $\Omega^\nabla = d^\nabla \omega \in \Omega^2(M, \text{End}(E))$

Bianchi identities

- algebraic Bianchi identity: $d^\nabla d^\nabla \alpha = \Omega^\nabla \wedge \alpha$
  - unlike d, not nilpotent in general
- differential Bianchi identity: $d^{\nabla^{\text{End}}} \Omega^\nabla = 0$
  - more generally, $d^{\nabla^{\text{End}}} d^{\nabla^{\text{End}}} \beta = [\Omega^\nabla \wedge \beta] \quad \forall \beta \in \Omega^k(M, \text{End}(E))$

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Integration of Bundle-valued Forms

With a connection, curve integrals defined thru pullback

$$\overset{\nabla}{\int_\gamma} \alpha = \overset{\nabla}{\int_{[0,1]}} \gamma^* \alpha = \int_0^1 \mathcal{R}_{\gamma,t} \alpha_{\gamma(t)} \left(\dot{\gamma}(t)\right) dt \in E_{\gamma(0)},$$

- parallel transport everything back to initial point of curve

Extension to a k-form over a retractable region S easy too

- define homeomorphism φ from S to unit k-dim ball B
- given evaluation point v, define $\gamma_{v,p}$ as $\varphi^{-1}(\varphi(p)$——$\varphi(v))$
- then define $\mathcal{R}_p^{\nabla,\varphi_v} \in \mathrm{Hom}(E_p, E_v)$ as // transport along $\gamma_{v,p}$

$$\underset{\varphi_v}{\overset{\nabla}{\int_S}} \alpha = \int_S \mathcal{R}^{\nabla,\varphi_v} \alpha \in E_v$$

- *note:* the homeomorphism can be defined through a strong deformation retraction to point v, $\varphi_v : [0, 1] \times S \rightarrow S$

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Discrete Setup (Abstractly First)

Let a simplicial complex M be an orientable manifold

Discrete Vector Bundle (of rank r)?

- a collection of vector spaces $\{\mathbf{E}_{v_i}\}$ with $v_i \in V$ (i.e., a vector space per vertex) and $\dim(\mathbf{E}_{v_i}) = r$.

Section of Discrete Frame Bundle?

- a collection of frames $\{\mathbf{F}_{v_i}\}$ with $v_i \in V$ defining an "arbitrary" choice of frame for each vector space $\mathbf{E}_{v_i}$.

Discrete connection ∇ ?

- a collection of maps $\mathcal{R}_{ij} : (\mathbf{E}_{v_j}, \langle\cdot,\cdot\rangle_{v_j}) \rightarrow (\mathbf{E}_{v_i}, \langle\cdot,\cdot\rangle_{v_i})$, one for each oriented edge e_{ij} of M, with $\mathcal{R}_{ij} \circ \mathcal{R}_{ji} = \mathbf{Id}_{v_i}$
- parallel transport maps, encoded as matrices R_{ij} given $\{\mathbf{F}_{v_i}\}_i$
- connection 1-form $\boldsymbol{\omega}_{v_0 v_1} = R_{v_0 v_1} - \mathrm{Id}$
  - approx. of path-ordered matrix exponential

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Discrete Bundle-valued Forms I

Abstract definition, given an evaluation fiber…

Definition (Discrete (1,0)-tensor-valued $\ell-$form). A discrete vector-valued $\ell-$form $\boldsymbol{\alpha}$ on M is a collection of maps which, for each $\ell-$simplex σ and one of its vertices v, returns a vector in $\mathbf{E}_v$, i.e.,

$$\boldsymbol{\alpha}\colon \sigma \in \mathcal{M}^\ell, v \in \sigma \subset \mathcal{V}(M) \mapsto \boldsymbol{\alpha}(\sigma, v) \in \mathbf{E}_v, \tag{1}$$

such that if $\bar{\sigma}$ is the simplex σ with reversed orientation, one has $\boldsymbol{\alpha}(\bar{\sigma}, v) = -\boldsymbol{\alpha}(\sigma, v)$ for all $v \in \sigma$.

- assume for now that a discrete bundle-valued ℓ-form is defined through its values on *all simplex-vertex pairs*
- eventually, will be one vector in $\mathbf{E}_v$ per ℓ-simplex à la DEC

Discrete Bundle-valued Forms II

For discrete endomorphism-valued ℓ-forms?

Definition (Discrete $(1,1)-$tensor-valued $\ell-$form). A discrete $(1,1)-$tensor-valued $\ell-$form $\boldsymbol{\beta}$ on M is a collection of maps which, for each $\ell-$simplex σ and two of its vertices (w, the input (or *cut*) fiber, and v, the output (or *evaluation*) fiber), returns a homomorphism between $\mathbf{E}_v$ and $\mathbf{E}_w$, i.e.,

$$\boldsymbol{\beta}\colon \sigma \in \mathcal{M}^\ell, v \in \sigma, w \in \sigma \mapsto \boldsymbol{\beta}(\sigma, v, w) \in \mathrm{Hom}(\mathbf{E}_w, \mathbf{E}_v), \tag{1}$$

such that if $\bar{\sigma}$ is the simplex σ with reversed orientation, one has $\boldsymbol{\beta}(\bar{\sigma}, v, w) = -\boldsymbol{\beta}(\sigma, v, w)$ for all $v, w \in \sigma$.

- for now, assume that this type of ℓ-form is defined through its values on *all simplex-vertex-vertex triplets*
- wait a bit to get a better understanding of this cut fiber…

Integration à la DEC?

Could the bundle-valued case be an extension of DEC?

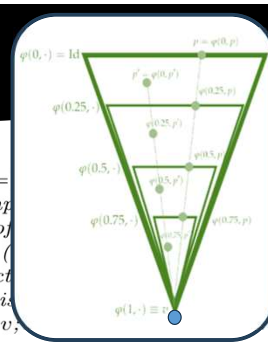
$$\int_S d^\nabla \alpha = \int_S d\alpha + \int_S \omega \wedge \alpha = \int_{\partial S} \alpha + \int_S \omega \wedge \alpha.$$

ANNOYING TERM IN MANY ASPECTS...

- can leverage choice of frame field to *bound* this term!
- pick a **frame field that makes ω zero *somewhere* in S**
  - will make the integration mostly about Stokes!
  - more precisely, $\mathcal{O}(h^{\ell+2})$ for an ℓ form on an (ℓ+1)-simplex

There is hope that a *discrete bundle-valued exterior calculus* can be built out of *discrete forms*, where the *integrals* of their smooth counterparts are evaluated using a **parallel-propagated frame field.**

Parallel-Propagated Frame



Definition (Continuous Parallel-Propagated Frame)
For a vector bunddle $\pi : E \to M$ with connection ∇, let $s=$ ... region for which there exists a diffeomorphism to an $\ell-$simp... with $v_i \mapsto w_i \forall i$. Let f be a local, arbitrary frame field o... s. For any given corner $v \in \{v_0, \ldots, v_\ell\}$, we also define a (... retraction $\varphi_v : [0,1] \times s \to s$ derived from a canonical retract... σ of the simplex σ through the aforementioned diffeomorphi... paths are radially joining the vertex w associated to point v,

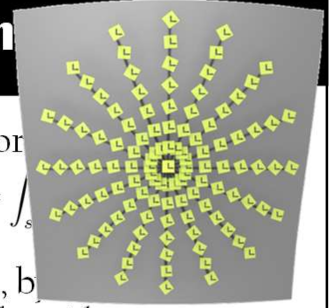
$$\varphi^\sigma_w : [0,1] \times \sigma \to \sigma$$
$$(t, p) \mapsto t\, w + (1-t)\, p.$$

Moreover, for any point $p \in \sigma$, we denote by $\mathcal{R}^{\nabla,v}(p) : \mathbf{E}_p \to \mathbf{E}_v$ the $\nabla-$induced parallel transport map from $\mathbf{E}_v$ to $\mathbf{E}_p$ along the path induced by the retraction φ_v and $R^{\nabla,v} : \mathbb{R}^r \to \mathbb{R}^r$ the matrix field representing $\mathcal{R}^{\nabla,v}(.)$ expressed in f. Frame field $\{f_a^{\nabla,v}\}$ over s is ***parallel-propagated frame field*** *from v if*

$$\mathcal{R}^{\nabla,v}(p) f_a^{\nabla,v}(p) = f_a(v), \quad \text{for all } p \in S, \text{ for all } a = 1, \ldots, r.$$

i.e., frame $f_a(v)$ at v has been parallel-transported throughout s via ∇. Furthermore, we call $R^{\nabla,v}$ the gauge field *of the PPF from $v \in M$.*

Parallel-Propagated Fram



We can now define integration of a for

$$\underset{s}{\overset{\nabla}{_{\varphi v}}\!\!\int} \alpha = \int_s \mathcal{R}^{\nabla,v}\alpha = \int_s \mathcal{R}^{\nabla,v} f_a^{\nabla,v}(p)(\alpha^{\nabla,v})^a = \int_s$$

Note that after gauge transformation, b

$\omega^{\nabla,v} = R^{\nabla,v}(\omega - (R^{\nabla,v})^{-1}dR^{\nabla,v})(R^{\nabla,v})^{-1}$ *vanishes* at vertex *v*

- PPF "follows" the bundle along radial lines emanating from *v*
- so $\|\omega^{\nabla,v}\| = \mathcal{O}(h)$ if Ω is bounded, *h* being the diameter of *s*

Consequently, one has

$$\int_s (d^\nabla \alpha)^{\nabla,v} = \int_{\partial s} \alpha^{\nabla,v} + \mathcal{O}(h^{\ell+2})$$

- exterior covariant derivative of α over simplex approximated by PPF-based integrals of α over the boundary faces of *s*

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Discrete Exterior Covariant Derivative

Now discrete version of d^∇ of [BHS2021] makes sense:

$$\mathfrak{d}^{\nabla}\boldsymbol{\alpha}([v_0,\ldots,v_{\ell+1}],v_0)$$
$$:= \mathcal{R}_{0,1}\ \boldsymbol{\alpha}([v_1,\ldots,v_{\ell+1}],v_1)$$
$$+ \textstyle\sum_{i=1}^{\ell+1}(-1)^i\boldsymbol{\alpha}([v_0,\ldots,\hat{v}_i,\ldots,v_{\ell+1}],v_0)$$

- just boundary terms; opposite face needs //-transport to v_0
- this sided operator converges under refinement ($h \to 0$)
  - if $\boldsymbol{\alpha}$ evaluated in ppf....
- but $\mathfrak{d}^\nabla \circ \mathfrak{d}^\nabla$ doesn't; ouch, Bianchi ids not meaningful...

Same comments for endomorphism-valued variant

$$\mathfrak{d}^{\nabla}\boldsymbol{\beta}(\sigma,v_0,v_{\ell+1}) := \mathcal{R}_{01}\boldsymbol{\beta}(\sigma_{v_0},v_1,v_{\ell+1})$$
$$+ \textstyle\sum_{i=1}^{\ell}(-1)^i\boldsymbol{\beta}(\sigma_{v_i},v_0,v_{\ell+1})$$
$$+ (-1)^{\ell+1}\boldsymbol{\beta}(\sigma_{v_{\ell+1}},v_0,v_\ell)\mathcal{R}_{\ell,\ell+1}$$

$\sigma_v = [v_0,\ldots,\hat{v},\ldots,v_{\ell+1}]$

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Discrete Exterior Covariant Derivative

Idea: *sided* derivatives not as good as *centered* ones…
Averaging sided estimates can gain an order of accuracy!

Averaging operator simple with a connection:

$$\mathrm{Alt}^{\nabla}(\boldsymbol{\alpha})([v_0, \ldots, v_\ell], v_0) := \frac{1}{(\ell+1)!} \sum_{\tau \in S_{\ell+1}} \mathrm{sgn}(\tau) \mathcal{R}_{v_0, v_{\tau(0)}} \boldsymbol{\alpha}([v_{\tau(0)}, \ldots, v_{\tau(\ell)}], v_{\tau(0)})$$

Similarly for its endomorphism-valued variant

$$\mathrm{Alt}^{\nabla} \boldsymbol{\beta}([v_0, \ldots, v_\ell], v_0, v_\ell) = \frac{1}{(\ell+1)!} \sum_{\tau \in S_{\ell+1}} \Big(\tfrac{1+\mathrm{sgn}(\tau)}{2} \mathcal{R}_{v_0, v_{\tau(0)}} \boldsymbol{\beta}([v_{\tau(0)}, \ldots, v_{\tau(\ell)}], v_{\tau(0)}, v_{\tau(\ell)}) \mathcal{R}_{v_{\tau(\ell)}, v_\ell} + \tfrac{\mathrm{sgn}(\tau)-1}{2} \mathcal{R}_{v_0, v_{\tau(0)}} \boldsymbol{\beta}([v_{\tau(0)}, \ldots, v_{\tau(\ell)}], v_{\tau(0)}, v_{\tau(\ell)}) \mathcal{R}_{v_{\tau(\ell)}, v_{\tau(\ell-1)}} \mathcal{R}_{v_{\tau(\ell-1)}, v_\ell} \Big).$$

- note that we can prove: $\mathrm{Alt}^{\nabla} \boldsymbol{\Omega}^{\nabla}([abc], a, c) = \boldsymbol{\Omega}^{\nabla}([abc], a, c)$

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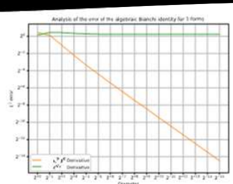
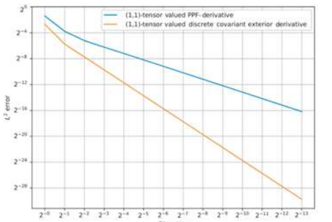
Discrete Exterior Covariant Derivative

So we propose a new discrete operator

$$\boldsymbol{d}^{\nabla} := \mathrm{Alt}^{\nabla} \eth^{\nabla}$$



- still satisfies all Bianchi identities at a discrete level
- both converge to their continuous counterparts
  - clear link to continuous case and $\boldsymbol{d}^{\nabla} \circ \boldsymbol{d}^{\nabla}$ converges



$$\beta = \begin{pmatrix} 0 & -xdy & 0 \\ xdy & 0 & dz \\ 0 & -dz & 0 \end{pmatrix} \in \Omega^1(\mathbb{R}^3, \mathrm{End}(T\mathbb{R}^3))$$

$$d^{\nabla^{\mathrm{End}}} \beta = \begin{pmatrix} 0 & -\,dx \wedge dy & y\, dx \wedge dz \\ dx \wedge dy & 0 & x^2 dy \wedge dz \\ -y\, dx \wedge dz & -x^2\, dy \wedge dz & 0 \end{pmatrix}.$$

$$\omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} x\, dz + \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (y\, dx + dz)$$

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Discrete Exterior Covariant Derivative

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- still satisfies all Bianchi identities at a discrete level
- both converge to their continuous counterparts
  - clear link to continuous case and $d^{\nabla} \circ d^{\nabla}$ converges
- algebraic Bianchi identity now reads

$$d^{\nabla} d^{\nabla} \alpha(\sigma, v_0) = \frac{1}{(\ell+3)!(\ell+2)!} \sum_{(m,\kappa)\in K} \Omega^{\nabla}(f, v_0, w_{m,\kappa})\, \alpha(\kappa, w_{m,\kappa})$$

$$=: \Omega^{\nabla} \wedge \alpha(s, v_0),$$

  - wedge product à la cup product

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Revisiting Curvature

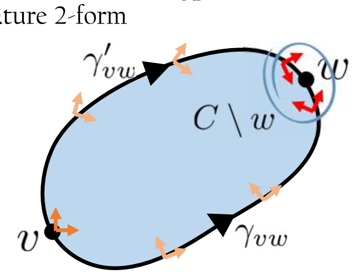
In the continuous case, $\Omega^{\nabla} = d\omega + \omega \wedge \omega$

In a PPF, we now get $\begin{cases} \tilde{\omega} = R\omega R^{-1} - dRR^{-1}. \\ \tilde{\Omega}^{\nabla} = R\Omega^{\nabla} R^{-1} = d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega}. \end{cases}$

- but in the PPF, $\tilde{\omega}(e_\rho)\tilde{\omega}(e_\theta) - \tilde{\omega}(e_\theta)\tilde{\omega}(e_\rho) = 0$ in $C \setminus w$, so

$$\int_C \tilde{\Omega}^{\nabla} = \int_C d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} = \int_{\partial C} \tilde{\omega} = \int_{\gamma_{vw}} \tilde{\omega} - \int_{\gamma'_{vw}} \tilde{\omega}.$$

  - mismatch at w is integral of curvature 2-form
  - extension of holonomy



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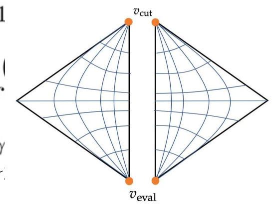
- but in the PPF, $\tilde{\omega}(e_\rho)\tilde{\omega}(e_\theta) - \tilde{\omega}(e_\theta)\tilde{\omega}($

  $$\int_C \widetilde{\Omega}^\nabla = \int_C d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} = \int_{\partial C} \tilde{\omega} = \int_\gamma$$

  - mismatch at w is integral of curvature 2-
  - extension of holonomy



For triangle *abc,* $\mathbf{\Omega}^\nabla(\sigma, a, c) = R_{ab}R_{bc} - R_{ac} \in \text{Hom}(\mathbf{E}_c, \mathbf{E}_a)$

- note indeed that it is $\boldsymbol{d}^\nabla \boldsymbol{\omega}$ since $\boldsymbol{\omega}_{ab} = R_{ab} - \text{Id}$
- shown to converge too in *O(h⁴)*
- advantage? can be summed!
  - matching evaluation and cut fibers implies matching retractions

Conclusions

Computation-ready exterior covariant derivatives

- structure preserving via discrete Bianchi identities
- converging to smooth equivalents in PPF
- for simplicial meshes for now – but extends to cell complexes

Did not talk about a few details…

- deRham and Whitney maps easy to formulate
- numerical tests require care
  - importance of path-ordered matrix exp, integrals thru quadratures,…
- in practice, we recommend using centroid-ppf, btw

Now what?

- except for Yang-Mills theory and relativity, is it useful?
  - revisiting elasticity and/or fluids, maybe…
- global structure of bundles satisfying Chern's characteristics?

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