Robust iterative splitting schemes for linear or nonlinear coupled PDEs (I and II)

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Joint work with J. Both, E. Storvik, K. Kumar, J. Stokke, M. Icardi, R. Nuca, I.S. Pop, K. Mitra, J. Kraus ...

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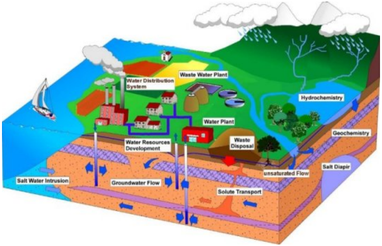
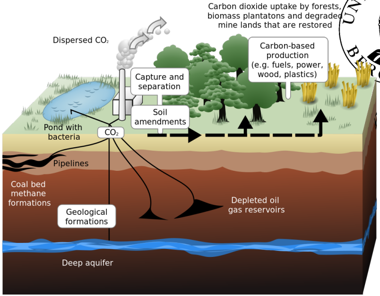
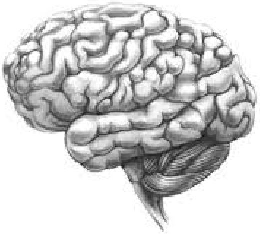
Overview

Coupled problems are everywhere!







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- **Combine the ingredients above to obtain robust and efficient schemes!**
- **We exemplify mostly for poromechanics, but the same ideas can be applied to e.g. phase field models** (Brun, Wick, Berre, Nordbotten, R., CMAME 2020; Storvik, Both, Sargado, Nordbotten, R., CMAME 2022), **surfactant transport** (Illiano, Pop, R., Comp. Geo. 2021), **Cahn-Larche or Cahn-Hilliard-Biot models** (Storvik, Both, Nordbotten, R., CAMWA 2023 or AML 2022) ...

The abstract problem

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Let V, W be Hilbert (Banach) spaces. Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$, $b(\cdot,\cdot) : W \times V \to \mathbb{R}$, $c(\cdot,\cdot) : W \times W \to \mathbb{R}$ be bilinear, $F_1 : V \to \mathbb{R}$, $F_2 : W \to \mathbb{R}$ be linear. The variational problem we want to solve reads:

Find $(u, p) \in V \times W$ s.t. there holds for all $(v, w) \in V \times W$

$$\left\{ \begin{array}{rcl} a(u,v) - b(p,v) & = & F_1(v), \\ c(p,w) + b(w,u) & = & F_2(w). \end{array} \right. \tag{1}$$

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Example poromechanics:

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &:= 2\mu\langle\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v})\rangle + \lambda\langle\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v}\rangle, \\ b(p,\mathbf{v}) &:= \langle p,\nabla\cdot\mathbf{v}\rangle, \\ c(p,w) &:= \frac{1}{M}\langle p,p\rangle + \tau k\langle\nabla p,\nabla p\rangle, \\ F_1(\mathbf{v}) &:= \langle\mathbf{f_1},\mathbf{v}\rangle, \\ F_2(w) &:= \langle\tau f_2 + p^{n-1} + \alpha\nabla\cdot\mathbf{u^{n-1}},\mathbf{w}\rangle \end{aligned}$$

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Plan:

- Set up a robust splitting scheme.
- Analysis: convergence.
- Example: fixed-stress splitting for poromechanics.
- Extensions:
  - Nonlinear equations.
  - Three or more coupled equations.

A robust splitting scheme

Let $(u^0, p^0) \in V \times W$ be given. For $i \geq 0$, find $(u^{i+1}, p^{i+1}) \in V \times W$ s.t. there holds for all $(v, w) \in V \times W$

$$\left\{ \begin{array}{rcl} a(u^{i+1}, v) - b(p^{i+1}, v) & = & F_1(v), \\ c(p^{i+1}, w) + L\langle p^{i+1} - p^i, w\rangle + b(w, u^i) & = & F_2(w). \end{array} \right. \tag{2}$$

Notation: i the iteration index.

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- $L \geq 0$ is a (free to chosen) tuning or stabilization parameter (it can be also a matrix or/and space/time dependent).
- Consistency.
- Uzawa algorithm.
- Anderson acceleration (has also a stabilization effect, see Both et al., CAMWA 2019).
- Convergence: To show that $e_p^i \to 0, e_u^i \to 0$ where $e_p^i := p - p^i, e_u^i := u - u^i$.
- Optimization.
- Analogously one can first solve for u and then for p, add a stabilization term to the first equation. The analysis is of course similar.

Notation: i the iteration index.

(Typical) Assumptions

- $a(\cdot,\cdot)$ is bilinear, symmetric, coercive ($a_\star$) and continuous (M_a).
- $b(\cdot,\cdot)$ is bilinear and continuous (M_b).
- $c(\cdot,\cdot)$ is bilinear, coercive ($c_\star$) and continuous (M_c).
- $F_i(\cdot)$, $i = 1, 2$ are linear.

Remark

• Not all the assumptions are necessarily! One or other coercivity constant can be even zero (this gives us the way to choose which splitting we use!).

Notation: n denotes the time level, i the iteration index, h the mesh diameter and τ the time step size.

Theorem

The splitting scheme (2) *converges linearly for any* $L \geq \frac{M_b^2}{2a_\star}$. *There holds*

$$\|e_p^{i+1}\| \leq \frac{L}{L+2c_\star}\|e_p^i\|, \qquad \text{and} \qquad \|\mathrm{e}_\mathrm{u}^\mathrm{i}\| \leq \mathrm{C}\|\mathrm{e}_\mathrm{p}^\mathrm{i}\|. \tag{3}$$

Remark

- We do not need the continuity of $a(\cdot,\cdot)$ and $c(\cdot,\cdot)$.
- It is a tricky proof, which can not be generalized to the nonlinear case.
- For $L < \frac{M_b^2}{2a_\star}$ the scheme may not converge, in this sense the result is sharp.
- The number of iterations depends on the choice of L. The smallest L is but not necessarily leading to optimal convergence!
- One can use the splitting scheme also to prove the existence of a solution for the problem (1). One introduces $\tilde{e}_p^i = p^{i+1} - p^i, \tilde{e}_u^i = u^{i+1} - u^i$.

Notation: i the iteration index, $e_p^i = p - p^i, e_u^i = u - u^i$.

Proof of the convergence

Notation: i the iteration index, $e_p^i = p - p^i, e_u^i = u - u^i$.
By subtracting the corresponding equations, we get the error equations

$$\begin{aligned} a(e_u^{i+1}, v) - b(e_p^{i+1}, v) &= 0, && (4) \\ c(e_p^{i+1}, w) + L\langle e_p^{i+1} - e_p^i, w\rangle + b(w, e_u^i) &= 0, && (5) \end{aligned}$$

for all $(v, w) \in V \times W$. Testing now the above with $v = e_u^i$ and $w = e_p^{i+1}$ and adding the resulting we obtain

$$a(e_u^{i+1}, e_u^i) + c(e_p^{i+1}, e_p^{i+1}) + L\langle e_p^{i+1} - e_p^i, e_p^{i+1}\rangle = 0. \qquad (6)$$

We recall now two important algebraic identities

$$\begin{aligned} a(u, v) &= \frac{1}{4}a(u+v, u+v) - \frac{1}{4}a(u-v, u-v) && (7) \\ (x-y, x) &= \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x-y\|^2 - \frac{1}{2}\|y\|^2. && (8) \end{aligned}$$

Proof of the convergence

Using now the identities (7) and (8) in (6) we get

$$\frac{1}{4}a(e_u^{i+1}+e_u^i, e_u^{i+1}+e_u^i) - \frac{1}{4}a(e_u^{i+1}-e_u^i, e_u^{i+1}-e_u^i) + c(e_p^{i+1}, e_p^{i+1}) \\ + \frac{L}{2}\|e_p^{i+1}\|^2 + \frac{L}{2}\|e_p^{i+1}-e_p^i\|^2 - \frac{L}{2}\|e_p^i\|^2 = 0.$$

Or, equivalently,

$$\frac{1}{4}a(e_u^{i+1}+e_u^i, e_u^{i+1}+e_u^i) + c(e_p^{i+1}, e_p^{i+1}) \\ + \frac{L}{2}\|e_p^{i+1}\|^2 + \frac{L}{2}\|e_p^{i+1}-e_p^i\|^2 = \frac{L}{2}\|e_p^i\|^2 + \frac{1}{4}a(e_u^{i+1}-e_u^i, e_u^{i+1}-e_u^i). \tag{9}$$

We come back to (4) to obtain for all $v \in V$

$$a(e_u^{i+1}-e_u^i, v) = b(e_p^{i+1}-e_p^i, v).$$

Testing the above with $v = e_u^{i+1} - e_u^i$ we get

$$a(e_u^{i+1}-e_u^i, e_u^{i+1}-e_u^i) = b(e_p^{i+1}-e_p^i, e_u^{i+1}-e_u^i). \tag{10}$$

Furthermore, by using the coercivity of $a(\cdot,\cdot)$ and the continuity of $b(\cdot,\cdot)$ we also get

$$\|e_u^{i+1}-e_u^i\| \le \frac{M_b}{a_\star}\|e_p^{i+1}-e_p^i\|. \tag{11}$$

Proof of the convergence

Using now the equations (10) and (11) in (9) we obtain

$$\begin{aligned}
\frac{1}{4}a(e_u^{i+1}+e_u^i, e_u^{i+1}+e_u^i) &+ c(e_p^{i+1}, e_p^{i+1}) \\
+\frac{L}{2}\|e_p^{i+1}\|^2 + \frac{L}{2}\|e_p^{i+1}-e_p^i\|^2 &\leq \frac{L}{2}\|e_p^i\|^2 + \frac{1}{4}b(e_p^{i+1}-e_p^i, e_u^{i+1}-e_u^i) \\
&\leq \frac{L}{2}\|e_p^i\|^2 + \frac{1}{4}M_b\|e_u^{i+1}-e_u^i\|\|e_p^{i+1}-e_p^i\| \\
&\leq \frac{L}{2}\|e_p^i\|^2 + \frac{M_b^2}{4a_\star}\|e_p^{i+1}-e_p^i\|^2. \qquad (12)
\end{aligned}$$

It is now easy to see that for $L \geq \frac{M_b^2}{2a_\star}$ we have a contraction

$$(\frac{L}{2}+c_\star)\|e_p^{i+1}\|^2 \leq \frac{L}{2}\|e_p^i\|^2.$$

This gives us the convergence of p^i to p. The convergence of u^i to u follows immediately.

Q.E.D.

Convergence analysis: alternative result

Theorem

The splitting scheme (2) *converges linearly for any* $L \geq \dfrac{M_b^2 M_a^2}{2a_\star^3}$. *There holds*

$$\|e_p^{i+1}\| \leq \frac{L}{L+2c_\star}\|e_p^i\|, \quad \text{and} \quad \|\mathrm{e}_\mathrm{u}^\mathrm{i}\| \leq \mathrm{C}\|\mathrm{e}_\mathrm{p}^\mathrm{i}\|. \tag{13}$$

Remark

- We use now the continuity of $a(\cdot,\cdot)$.
- The proof is relatively straightforward and can be generalized to the nonlinear case.
- The result is not sharp, obviously there holds $\dfrac{M_b^2 M_a^2}{2a_\star^3} \geq \dfrac{M_b^2}{2a_\star}$.
- The number of iterations depends on the choice of L. The smallest L is but not necessarily leading to optimal convergence!

Notation: i the iteration index, $e_p^i = p - p^i, e_u^i = u - u^i$.

The linear (quasi-static) Biot model

Mechanics:

$$-\nabla\cdot\Big[\underbrace{\underbrace{2\mu\varepsilon(\mathbf{u})+\lambda\nabla\cdot\mathbf{u}\,\mathbf{I}}_{\text{elastic stress}}-\alpha\underbrace{p}_{\text{pore pressure}}\mathbf{I}}_{\text{poroelastic stress}}\Big] = \mathbf{f}$$

$$\varepsilon(\mathbf{u}) := \frac{\nabla\mathbf{u}+\nabla\mathbf{u}^T}{2}$$

Flow:

$$\underbrace{1/M}_{\text{compressibility}}\,\partial_t p+\underbrace{\alpha\partial_t\nabla\cdot\mathbf{u}}_{\text{coupling}}+\nabla\cdot\mathbf{q} = 0$$

$$\mathbf{q} = -k\left(\nabla\mathbf{p}-\rho\mathbf{g}\right)$$

$\mathbf{u}$	displacement	$\mu,\ \lambda$	Lamé parameters	ρ	fluid density
p	fluid pressure	α	Biot coefficient	$1/M$	compr. coef.
$\mathbf{q}$	fluid flux	$\mathbf{I}$	identity tensor	k	mobility
$\varepsilon(\mathbf{u})$	linear strain	$\mathbf{f}$	volume force	$\mathbf{g}$	gravity

Extensions: nonlinear Biot, multiphase Biot, Biot-Allard

Discretization

Mechanics:

- Continuous Galerkin formulation $(\mathbf{V_h})$ – $\mathbb{P}_1$ or $\mathbb{P}_2$ finite elements for $\mathbf{u}$.

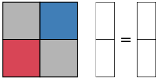
Flow:

- Mixed formulation $(W_h \times \mathbf{Q_h})$ – $\mathbb{RT}_0 \times \mathbb{P}_0$ mixed finite elements for $p, \mathbf{q}$ or continuous Galerkin (W_h) – $\mathbb{P}_1$.
- Implicit Euler time discretization.

Stabilization: Rodrigo, Hu, Ohm, Adler, Gaspar, Zikatanov, CMAME, 2018.

Notation: n denotes the time level, i the iteration index, h the mesh diameter and τ the time step size.

Iterative schemes for solving the linear Biot model



– Monolithic:

$$\begin{bmatrix} \frac{1}{M}p^n + \alpha\nabla\cdot\mathbf{u^n} + \tau\nabla\cdot\mathbf{q^n} \\ k\nabla\mathbf{p^n} + \mathbf{q^n} \end{bmatrix} = \mathsf{Res}_{flow}(\mathbf{u^{n-1}}, \mathbf{p^{n-1}})$$

$$-\nabla\cdot\left[\mathbf{2}\mu\varepsilon\left(\mathbf{u^n}\right) + \lambda\nabla\cdot\mathbf{u^n I} - \alpha\mathbf{p^n I}\right] = f$$

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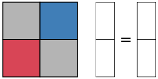

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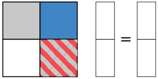

– Fixed-Stress Splitting (Settari, Mourits, SPE, 1998; Kim, Tchelepi, Juanes, CMAME, 2011;...):

Freeze mean stress: $\sigma_v^{n,i} = K_{dr}\nabla\cdot\mathbf{u^{n,i}} - \alpha\mathbf{p^{n,i}} = \mathbf{0} \quad \Leftrightarrow \quad \alpha\nabla\cdot\mathbf{u^{n,i}} = \frac{\alpha^2}{\mathbf{K_{dr}}}\mathbf{p^{n,i}}$

Solve:

$$1\begin{bmatrix} \frac{1}{M}p^{n,i+1} + \alpha\nabla\cdot\mathbf{u^{n,i}} + \mathbf{L}(\mathbf{p^{n,i+1}} - \mathbf{p^{n,i}}) + \tau\nabla\cdot\mathbf{q^{n,i+1}} \\ k\nabla\mathbf{p^{n,i+1}} + \mathbf{q^{n,i+1}} \end{bmatrix} = \text{Res}_{flow}(\mathbf{u^{n-1}}, \mathbf{p^{n-1}})$$

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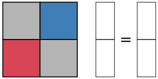

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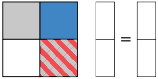

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Solve:

1: $$\begin{bmatrix} \frac{1}{M}p^{n,i+1} + \alpha\nabla\cdot\mathbf{u^{n,i}} + \mathbf{L}(\mathbf{p^{n,i+1}} - \mathbf{p^{n,i}}) + \tau\nabla\cdot\mathbf{q^{n,i+1}} \\ k\nabla\mathbf{p^{n,i+1}} + \mathbf{q^{n,i+1}} \end{bmatrix} = \mathrm{Res}_{flow}(\mathbf{u^{n-1}}, \mathbf{p^{n-1}})$$

2: $$-\nabla\cdot\left[\mathbf{2}\mu\varepsilon\left(\mathbf{u^{n,i+1}}\right) + \lambda\nabla\cdot\mathbf{u^{n,i+1}}\mathbf{I} - \alpha\mathbf{p^{n,i+1}}\mathbf{I}\right] = f$$

Fixed-Stress Splitting – Analysis

Convergence of the fixed-stress splitting scheme

$$L_{phy} = \frac{\alpha^2}{K_{dr}}, L_{min} = \frac{\alpha^2}{2K_{dr}} = \frac{L_{phy}}{2}, \quad K_{dr} = \frac{2\mu}{d} + \lambda.$$

- The fixed-stress scheme is linearly convergent for any $L \geq L_{min}$ w. r. t. $\|\cdot\|_\star$ for homogeneous media [Mikelić, Wheeler, Comput Geosci 2014]:

$$\|e_p^i, e_{\boldsymbol{u}}^i\|_\star^2 \leq \frac{L}{c+L}\|e_p^{i-1}, e_{\boldsymbol{u}}^{i-1}\|_\star^2.$$

- The fixed-stress scheme is linearly convergent for any $L \geq L_{min}$ w. r. t. $\|\cdot\|$ for:
  •• **heterogeneous media** [Both, Borregales, Nordbotten, Kumar, R., AML 2017]
  •• **higher-order space-time elements** [Bause, R., Köcher, R., CMAME 2017]
  •• **a partially parallel-in-time setting** [Borregales, Kumar, R., Rodrigo, Gaspar, Comp. & Math. with Appl. 2019]
  •• **a moving boundary poromechanic pb.** [Cerroni, Zunino, R., Math. in Engineering 2019]

$$\|e_p^i\|^2 \leq \frac{L}{c+L+\tau c(\Omega,k)}\|e_p^{i-1}\|^2, \quad \|e_{\boldsymbol{u}}^i\| \lesssim \|e_p^i\|.$$

- Similar results are obtained for the undrained split scheme.

Notation: $e_p^i = p^n - p^{n,i}, e_{\boldsymbol{u}}^i = \boldsymbol{u}^n - \boldsymbol{u}^{n,i}$, $\|\cdot\|$ is the L^2-norm.

Fixed-Stress Splitting – Optimization

Optimization of the stabilization/tuning parameter L

Question: By known K_{dr}, which L gives the lowest number of iterations (in this sense being optimal)

$$L_{phy} = \frac{\alpha^2}{K_{dr}} \text{ or } \mathrm{L}_{\min} = \frac{\mathrm{L}_{\mathrm{phy}}}{2}?$$

Assumption (A1). There exists $\beta > 0$ such that for any $p \in W_h$ there exists $\mathbf{u} \in \mathbf{V}_h$ satisfying $\nabla \cdot \mathbf{u} = p$ and

$$2\mu\|\varepsilon(\mathbf{u})\|^2 + \lambda\|\nabla \cdot \mathbf{u}\|^2 \leq \beta\|p\|^2. \quad (14)$$

• **Theorem** [Storvik, Both, Kumar, Nordbotten, R., IJNME 2019.] Assume that (A1) holds true. The fixed-stress splitting scheme converges linearly for $L \geq L_{min}$. The optimal stabilization parameter is

$$L := \frac{\alpha^2}{\mathcal{D}K_{dr}} \in [L_{min}, L_{phy}], \text{ where } \mathcal{D} := \min\left\{\frac{\mathrm{A}}{2\mathrm{B}}, 2\right\} \in [1, 2], \quad (15)$$

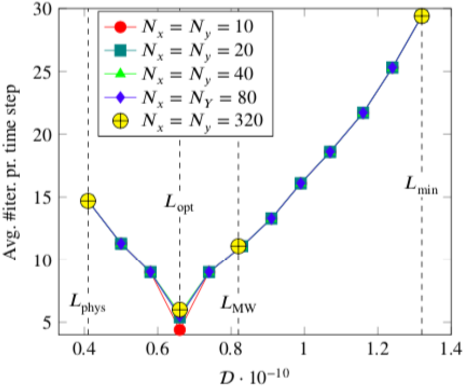
$A := \dfrac{2}{M} + \dfrac{2\tau\kappa}{C_\Omega^2} + \dfrac{2\alpha^2}{\beta}$, $B := \dfrac{\alpha^2}{\beta}$, C_Ω denotes the Poincaré constant and β is the constant from (14).

Fixed-Stress Splitting – Practical optimization

Practical optimization of the stabilization/tuning parameter L

- Question: How do we get practically the optimal stabilization parameter L?
- Let us summarise what we just learned:
  - The optimal L lies in the interval $[L_{min}, L_{phy}]$.
  - It does not depend on the mesh size.
- Algorithm:
  - Consider a coarse mesh.
  - Consider an equidistant partition of the interval $[L_{min}, L_{phy}]$ in e.g. 10 intervals.
  - Perform simulations with each of the numbers in the above partition (for 10 intervals will be 11 numbers) for the coarse mesh and one time step.
  - Choose the optimal L_{opt} to be the one which gives the lowest number of iterations.
- Details: Storvik, Both, Kumar, Nordbotten, R., IJNME 2019.

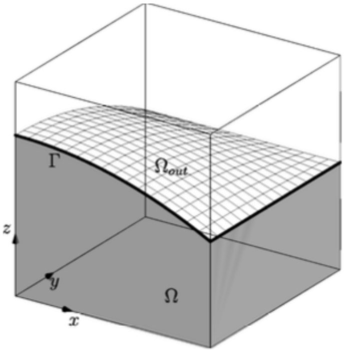
Numerical results: optimal stabilization parameter



- Mandel's problem: Number of iterations for different stabilization parameters, $L = \dfrac{\alpha^2}{\mathcal{D}}$.
- Details: Storvik, Both, Kumar, Nordbotten, R., IJNME 2019.

A poroelastic problem with a moving boundary

- Deformation of a sedimentary basin undergoing surface erosion.
- The movement of the boundary is given by the zero value of a level set function.
- The mathematical model is given by Biot equations.
- To avoid remeshing we use **CutFEM** (Burman et al, IJNME 2015).
- We use fixed-stress splitting (see e.g. Both et al, AML, 2017).



A poroelastic problem with a moving boundary

• **Problem**: the appearance of small size elements leading to ill-conditioned systems and large number of fixed-stress iterations.

• **Remedy**:

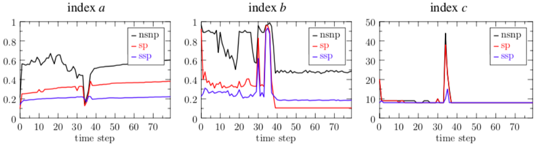
  – Stabilization (use of a ghost penalty stabilization operator, see Burman, Hansbo, Larson, Math Comp 2018, Burman, Zunino, Springer 2012).

  – Preconditioning.

• **A theoretical analysis** is performed concerning the stability, condition number of the linear systems and convergence of the fixed-stress splitting.

• **Details**: Cerroni, R., Zunino, Numerical solvers for a poromechanic problem with a moving boundary, Mathematics in Engineering, 2019.

Numerical results: 3D test problem, P2-P1 discretization



(a) averaged nb. of GMRES iterations for pressure / max nb. of GMRES iterations

(b) averaged nb. of GMRES iterations for displacement / max nb. of GMRES iterations

(c) nb. of fixed-stress iterations

(*nsnp*) non stabilized, non preconditioned (black line); it refers to the original formulation of the problem.

(*sp*) pressure-stabilized and preconditioned (red line); it refers to the stabilized formulation of the pressure problem and the preconditioner P for the elasticity problem.

(*ssp*) pressure and displacement stabilized and preconditioned (blue line); these data are are obtained using the ghost penalty stabilization $j_u(\cdot,\cdot)$ in the elasticity problem.

Schur complement based splitting schemes (Nuca, Storvik, R., Icardi, CAMWA 2024)

Solve the system:

$$\begin{cases} Au + Bv &= f_1 \\ Cu + Dv &= f_2 \end{cases}$$

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$$\begin{cases} Au + Bv &= f_1 \\ Cu + Dv &= f_2 \end{cases}$$

Applications:

- **Dual-Porosity model**

$$\begin{cases} -\nabla \cdot (m_u \nabla u) &= \beta(v-u) \\ -\nabla \cdot (m_v \nabla v) &= \beta(u-v) \end{cases}$$

- **Quad-Laplacian model**

$$\begin{cases} -\nabla \cdot (m_{uu} \nabla u) - \nabla \cdot (m_{uv} \nabla v) &= f_1 \\ -\nabla \cdot (m_{vu} \nabla u) - \nabla \cdot (m_{vv} \nabla v) &= f_2 \end{cases}$$

- **Poromechanics**
- ...

Idea: write $A = \tilde{A} + (A - \tilde{A})$. From $Au + Bv = f_1$ one simply gets

$$u = \tilde{A}^{-1} f_1 - \tilde{A}^{-1}(A - \tilde{A})u - \tilde{A}^{-1} Bv.$$

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Using the above in $Cu + Dv = f_2$ we obtain

$$(D - C\tilde{A}^{-1}B)v = f_2 - C\tilde{A}^{-1} f_1 + C\tilde{A}^{-1}(A - \tilde{A})u.$$

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In this way we get the splitting scheme: Let u^0, v^0 be arbitrarly. For $k \geq 0$, find u^{k+1}, v^{k+1} such that

$$\begin{cases} (D - C\tilde{A}^{-1}B)v^{k+1} & = \quad f_2 - C\tilde{A}^{-1} f_1 + C\tilde{A}^{-1}(A - \tilde{A})u^k \\ Au^{k+1} + Bv^{k+1} & = \quad f_1. \end{cases}$$

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Relation to stabilized splitting schemes. The scheme above is equivalent to (use $Au^k + Bv^k = f_1$): Let u^0, v^0 be arbitrarly. For $k \geq 0$, find u^{k+1}, v^{k+1} such that

$$\begin{cases} Dv^{k+1} + Cu^k + C\tilde{A}^{-1}B(v^k - v^{k+1}) &= f_2 \\ Au^{k+1} + Bv^{k+1} &= f_1. \end{cases}$$

It is a stabilized splittting scheme with a non-diagonal $L = -C\tilde{A}^{-1}B$!

One can obtain similarly other splitting schemes by writing $D = \tilde{D} + (D - \tilde{D})$, $C = \tilde{C} + (C - \tilde{C})$ or $B = \tilde{B} + (B - \tilde{B})$. Example:

Let u^0, v^0 be arbitrarly. For $k \geq 0$, find u^{k+1}, v^{k+1} such that

$$\begin{cases} (A - B\tilde{D}^{-1}C)u^{k+1} &= f_1 - B\tilde{D}^{-1}(f_2 - (D - \tilde{D})v^k \\ (D - C\tilde{A}^{-1}B)v^{k+1} &= f_2 - C\tilde{A}^{-1}(f_1 - (A - \tilde{A})u^{k+1}). \end{cases}$$

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• One can do numerical computations to compare the different schemes for each problem.

• More details: Nuca, Storvik, R., Icardi, CAMWA 2024.

Splitting schemes for two coupled nonlinear equations

Let V, W be Hilbert (Banach) spaces. Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$, $b(\cdot,\cdot) : W \times V \to \mathbb{R}$, $c(\cdot,\cdot) : W \times W \to \mathbb{R}$ may be nonlinear, $F_1 : V \to \mathbb{R}$, $F_2 : W \to \mathbb{R}$ be linear. The variational problem we want to solve reads:

Find $(u, p) \in V \times W$ s.t. there holds for all $(v, w) \in V \times W$

$$\begin{cases} a(u,v) - b(p,v) &= F_1(v), \\ c(p,w) + b(w,u) &= F_2(w). \end{cases} \tag{16}$$

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What we need?

- **Linearization**
- **Splitting**
- **Stabilization**
- **Acceleration**
- Optimization

$L-$scheme or stabilized Picard scheme (Pop, R., Knabner, JCAM 2004, List, R., Comp. Geo. 2016, Mitra, Pop, CAMWA 2019 (modified L-scheme) ...)

Solve:

Nonlinear, nonsmooth equation: Find p such that $b(p) = r$.

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Solve:

Nonlinear, nonsmooth equation: Find p such that $b(p) = r$.

Idea:

Quasi-Newton method employing a stabilization parameter $L > 0$:

$$r = b(p_w^{i+1}) \approx b(p_w^i) + b'(p_w^i)\,(p_w^{i+1} - p_w^i) \qquad \text{[Newton]}$$

$$\approx b(p_w^i) + L\,(p_w^{i+1} - p_w^i) \qquad \text{[L-scheme]}$$

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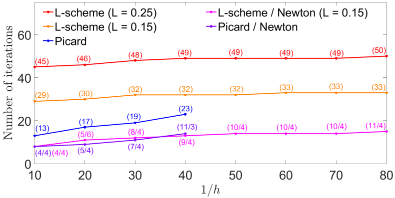
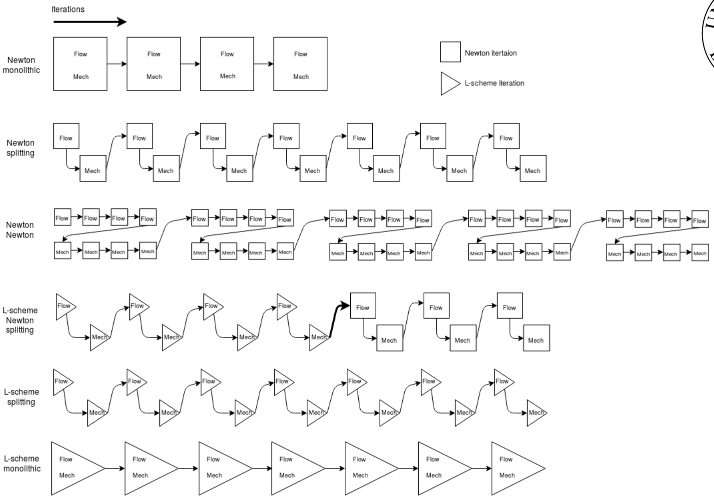
Main features:

- Does not involve the computation of derivatives.
- The linear systems are very well-conditioned.
- **Adaptive Newton/L-scheme** in Stokke, Mitra, Storvik, Both, R., CAMwA 2023.
- Non-Lipschitz nonlinearities allowed [R., Kumar, Pop, Nordbotten, IMA J. Num. Anal. 2018].



Solving strategies for nonlinear poromechanics: schemes



- The schemes can be applied in combination with any spatial discretization.
- We implemented them also for higher-order space-time elements.

A robust splitting scheme

Let $(u^0, p^0) \in V \times W$ be given. For $i \geq 0$, find $(u^{i+1}, p^{i+1}) \in V \times W$ s.t. there holds for all $(v, w) \in V \times W$

$$\left\{ \begin{array}{rcl} a(u^{i+1}, v) - b(p^{i+1}, v) + L_2 \langle u^{i+1} - u^i, v \rangle & = & F_1(v), \\ c(p^{i+1}, w) + L_1 \langle p^{i+1} - p^i, w \rangle + b(w, u^i) & = & F_2(w). \end{array} \right. \tag{17}$$

Notation: i the iteration index.

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- $L_1, L_2 \geq 0$ are (free to chosen) tuning or stabilization parameters. We need two now!
- Consistency.
- Anderson acceleration.
- Convergence: To show that $e_p^i \to 0, e_u^i \to 0$ where $e_p^i := p - p^i, e_u^i := u - u^i$.
- Optimization
- One can use the splitting scheme also to prove the existence of a solution for the problem (1). One introduces $\tilde{e}_p^i = p^{i+1} - p^i, \tilde{e}_u^i = u^{i+1} - u^i$

Notation: i the iteration index.

Different nonlinear Biot models

• Nonlinear parameters [Borregales, Radu, Kumar, Nordbotten, Comput Geosci 2018] Assuming that the nonlinearities are Lipschitz continuous and monotone increasing, and that the stabilization parameters are big enough, then all the considered schemes are at least linearly convergent. The only quadratic convergent scheme is the monolithic Newton.
• Large deformations case [Borregales, Kumar, Nordbotten, Radu, Comput Geosci 2020] Difficult to prove something theoretically. Many different schemes were (numerically) tested. Convincing numerical results.
• Saturated/unsaturated flow in deformable porous media [Both, Kumar, Nordbotten, Radu, CAMWA 2019]. Anderson acceleration (stabilizing effect!).
• Thermo-poroelasticity [Brun, Ahmed, Berre, Nordbotten, Radu, CAMWA 2020]. Convergence can be proven theoretically. Different combinations were considered.
• Soft material poromechanics [Both, Barnafi, R., Zunino, Quarteroni, CMAME 2021]. A very complex model. Convergence can be proven theoretically.

Example 1: Nonlinear Lamé parameters and compression

A first nonlinear (quasi-static) Biot model

Mechanics:

$$-\nabla \cdot \underbrace{\Big[\underbrace{2\mu\varepsilon(\mathbf{u}) + \lambda(\nabla \cdot \mathbf{u})\,\mathbf{I}}_{\text{elastic stress}} - \alpha \underbrace{p}_{\text{pore pressure}} \mathbf{I}\Big]}_{\text{poroelastic stress}} = \mathbf{f}$$

$$\varepsilon(\mathbf{u}) := \frac{\nabla\mathbf{u} + \nabla\mathbf{u}^T}{2}$$

Flow:

$$\underbrace{1/M}_{\text{compressibility}}\,\partial_t b(p) + \underbrace{\alpha\partial_t \nabla \cdot \mathbf{u}}_{\text{coupling}} + \nabla \cdot \mathbf{q} = 0$$

$$\mathbf{q} = -k\,(\nabla \mathbf{p} - \rho\mathbf{g})$$

$\mathbf{u}$	displacement	$\mu,\ \lambda$	Lamé parameters	ρ	fluid density
p	fluid pressure	α	Biot coefficient	$1/M$	compr. coef.
$\mathbf{q}$	fluid flux	$\mathbf{I}$	identity tensor	k	mobility
$\varepsilon(\mathbf{u})$	linear strain	$\mathbf{f}$	volume force	$\mathbf{g}$	gravity

Numerical schemes

A (fixed-stress type) splitting $L-$scheme

Step 1:

$$\begin{aligned} b(p^{n,i}) + L_1(p^{n,i+1} - p^{n,i}) + \alpha\nabla\cdot(\boldsymbol{u}^{n,i} - \boldsymbol{u}^{n-1}) + \tau\nabla\cdot\boldsymbol{q}^{n,i+1} &= \tau S_f + b(p^{n-1}) \\ \boldsymbol{q}^{n,i+1} + k\nabla p^{n,i+1} &= \rho_f \boldsymbol{g} \end{aligned}$$

Step 2:

$$-\nabla\cdot[2\mu\varepsilon(\boldsymbol{u}^{n,i+1}) + h(\nabla\cdot\boldsymbol{u}^{n,i}) + L_2(\boldsymbol{u}^{n,i+1} - \boldsymbol{u}^{n,i}) - \alpha p^{n,i+1} I] = \boldsymbol{f}$$

- Initialize $p^{n,0} = p^{n-1}$, $\boldsymbol{q}^{n,0} = \boldsymbol{q}^{n-1}$, $\boldsymbol{u}^{n,0} = \boldsymbol{u}^{n-1}$.
- i is the iteration step and n is the time level.

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\end{aligned}
$$

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$$

A Newton splitting scheme

Step 1:

$$
\begin{aligned}
b(p^{n,i}) + b'(p^{n,i})(p^{n,i+1} - p^{n,i}) + \alpha\nabla\cdot(\boldsymbol{u}^{n,i} - \boldsymbol{u}^{n-1}) + \tau\nabla\cdot\boldsymbol{q}^{n,i+1} &= \tau S_f + b(p \\
\boldsymbol{q}^{n,i+1} + k\nabla p^{n,i+1} &= \rho_f \boldsymbol{g}
\end{aligned}
$$

Step 2:

$$
-\nabla\cdot[2\mu\varepsilon(\boldsymbol{u}^{n,i+1}) + h(\nabla\cdot\boldsymbol{u}^{n,i}) + (L_2 + h'(\nabla\cdot\boldsymbol{u}^{n,i}))(\boldsymbol{u}^{n,i+1} - \boldsymbol{u}^{n,i}) - \alpha p^{n,i+1} I] = \boldsymbol{f}
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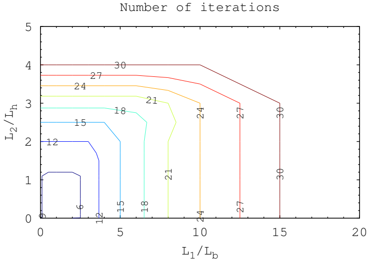
Convergence analysis

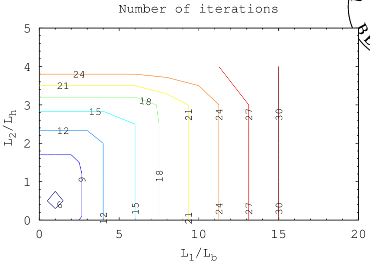
Theorem [Borregales, R., Kumar, Nordbotten, Comput Geosci 2018] Assuming that the nonlinearities are Lipschitz continuous and monotone increasing, and that the stabilization parameters are big enough, then all the considered schemes are at least linearly convergent. The only quadratic convergent scheme is the monolithic Newton.

Remarks

• The only quadratic convergent scheme (but not robust) is the monolithic Newton scheme.
• The splitting and monolithic schemes have very similar properties.
• The splitting Newton schemes require a stabilization term.
• For extensions to non-Lipschitz nonlinearities we refer to Both, Kumar, Nordbotten, Pop, R., Springer, 2019.

Numerical results



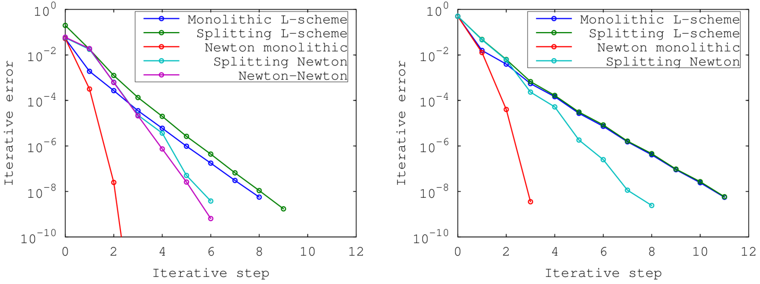


Splitting

Monolithic

Performance of the schemes for different values of L_1 and L_2 for test problem 1,
$b(p) = e^p; \;\; h(\nabla \cdot \boldsymbol{u}) = (\nabla \cdot \boldsymbol{u})^3.$

Newton method convergence: 5 iterations.



Convergence at a given time and on a fixed mesh for the different schemes. Nonlinearities: left $b(p) = p^2$, $h(\nabla \cdot \boldsymbol{u}) = (\nabla \cdot \boldsymbol{u})^2$, right $b(p) = e^p$, $h(\nabla \cdot \boldsymbol{u}) = \sqrt[3]{(\nabla \cdot \boldsymbol{u})^5} + \nabla \cdot \boldsymbol{u}$.

Saturated/unsaturated flow and deformation: Mathematical model

Mechanics:
$$-\nabla \cdot [\mathbf{2}\mu\varepsilon(\mathbf{u}) + \lambda\nabla\cdot\mathbf{u}\,\mathbf{I} - \alpha\mathbf{p}_{pore}(\mathbf{s_w},\mathbf{p_w})\,\mathbf{I}] = \rho_\mathbf{b}\mathbf{g}$$

Flow:
$$\phi_0\partial_t s_w + \alpha s_w\partial_t\nabla\cdot\mathbf{u} + \nabla\cdot\mathbf{q_w} = 0$$
$$\mathbf{q_w} + \mathbf{k_w}(\nabla\mathbf{p_w} - \rho_\mathbf{w}\mathbf{g}) = \mathbf{0}$$

$\mathbf{u}$	displacement	$s_w = s_w(p_w)$	water saturation
p_w	water pressure	$k_w = k_w(s_w)$	water mobility
$\mathbf{q_w}$	water flux	$\varepsilon(\mathbf{u})$	linear strain

Pore pressure [Coussy, 2004]:

$$p_{pore} = \chi p_w + (1-\chi)\underbrace{p_{nw}}_{=0} = s_w p_w - \int_{s_w}^{1} p_c(s)ds$$

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Pore pressure [Coussy, 2004]:

$$p_{pore}=\chi p_w+(1-\chi)\underbrace{p_{nw}}_{=0}=s_w p_w-\int_{s_w}^{1}p_c(s)ds$$

Aim: Robust and fast numerical scheme!

Saturated/unsaturated flow and deformation: Mathematical model

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$\mathbf{q_w}$	water flux	$\varepsilon(\mathbf{u})$	linear strain

Pore pressure [Coussy, 2004]:

$$p_{pore} = \chi p_w + (1-\chi)\underbrace{p_{nw}}_{=0} = s_w p_w - \int_{s_w}^{1} p_c(s)\,ds$$

Aim: Robust and fast numerical scheme!

We can set in a similar way monolithic and splitting schemes based on Newton, L-scheme or modified Picard linearizations (and combination of them).

A monolithic $L-$scheme for unsaturated flow and mechanics

Nonlinear Biot's equations:

$$
\begin{aligned}
\phi_0 \partial_t s_w + \alpha s_w \partial_t \nabla \cdot \mathbf{u} + \nabla \cdot \mathbf{q_w} &= 0 \\
\mathbf{q_w} + \mathbf{k_w} \left(\nabla \mathbf{p_w} - \rho_{\mathbf{w}} \mathbf{g} \right) &= \mathbf{0} \\
-\nabla \cdot \left[\mathbf{2} \mu \varepsilon \left(\mathbf{u} \right) + \lambda \nabla \cdot \mathbf{u} \, \mathbf{I} - \alpha \mathbf{p}_{pore}(\mathbf{s_w}, \mathbf{p_w}) \, \mathbf{I} \right] &= \rho_b \mathbf{g}
\end{aligned}
$$

A monolithic $L-$scheme for unsaturated flow and mechanics

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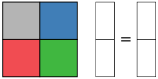
\end{aligned}
$$



– Monolithic $L-$scheme:

$$
\begin{bmatrix}
s_w^{n,i} + L(p_w^{n,i+1} - p_w^{n,i}) + \alpha s_w^{n,i} \nabla \cdot \mathbf{u^{n,i+1}} + \tau \nabla \cdot \mathbf{q_w^{n,i+1}} \\
k_w^{n,i} \nabla \mathbf{p_w^{n,i+1}} + \mathbf{q_w^{n,i+1}}
\end{bmatrix}
= \mathrm{Res}_{flow}
$$

$$
-\nabla \cdot \left[\mathbf{2}\mu\varepsilon \left(\mathbf{u^{n,i+1}} \right) + \lambda \nabla \cdot \mathbf{u^{n,i+1}} \mathbf{I} - \alpha \mathbf{s_w^{n,i}} \mathbf{p_w^{n,i+1}} \mathbf{I} \right] = \mathrm{Res}_{mech}
$$

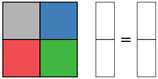
A splitting scheme for unsaturated flow and mechanics

– Monolithic $L-$scheme:

$$\begin{bmatrix} s_w^{n,i} + L(p_w^{n,i+1} - p_w^{n,i}) + \alpha s_w^{n,i} \nabla \cdot \mathbf{u^{n,i+1}} + \tau \nabla \cdot \mathbf{q_w^{n,i+1}} \\ k_w^{n,i} \nabla \mathbf{p_w^{n,i+1}} + \mathbf{q_w^{n,i+1}} \end{bmatrix} = \mathrm{Res}_{flow}$$

$$-\nabla \cdot \left[\mathbf{2}\mu\varepsilon\left(\mathbf{u^{n,i+1}}\right) + \lambda \nabla \cdot \mathbf{u^{n,i+1} I} - \alpha \mathbf{s_w^{n,i} p_w^{n,i+1} I} \right] = \mathrm{Res}_{mech}$$

A splitting scheme for unsaturated flow and mechanics



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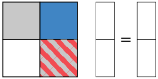


– Splitting scheme:

$$\begin{bmatrix} s_w^{n,i} + L(p_w^{n,i+1} - p_w^{n,i}) + \alpha s_w^{n,i} \nabla \cdot \mathbf{u^{n,i}} + \tau \nabla \cdot \mathbf{q_w^{n,i+1}} \\ k_w^{n,i} \nabla \mathbf{p_w^{n,i+1}} + \mathbf{q_w^{n,i+1}} \end{bmatrix} = \mathsf{Res}_{flow}$$

$$-\nabla \cdot \left[\mathbf{2}\mu\varepsilon\left(\mathbf{u^{n,i+1}}\right) + \lambda \nabla \cdot \mathbf{u^{n,i+1}} \mathbf{I} - \alpha \mathbf{s_w^{n,i}} \mathbf{p_w^{n,i+1}} \mathbf{I} \right] = \mathsf{Res}_{mech}$$

Acceleration possibilities [Both, Kumar, Nordbotten, R., CAMWA, 2019]

Combine the $L-$scheme and Newton:

- Use $L-$scheme to obtain a sufficiently good guess for Newton's method.
- Strategy works for Richards' equation [List and R., Comput. Geo., 2016].

Acceleration possibilities [Both, Kumar, Nordbotten, R., CAMWA, 2019]

Combine the $L-$scheme and Newton:

- Use $L-$scheme to obtain a sufficiently good guess for Newton's method.
- Strategy works for Richards' equation [List and R., Comput. Geo., 2016].

Use the Anderson acceleration for fixed-point iterations [Anderson, 1965, Walker and Ni, SINUM 2011]

- It accelerates the convergence.
- It has also a stabilization effect, and can be used in combination with the Newton method to increase its robustness.
- We proved theoretically for a simplified situation that Anderson acceleration makes a non-contractive fixed-point scheme convergent [Both, Kumar, Nordbotten, R., CAMWA, 2019].

Anderson acceleration for fixed-point iterations [Anderson, 1965, Walker and Ni, SINUM 2011]

Initial situation:

L-scheme combined with fixed-stress splitting has fixed point type character

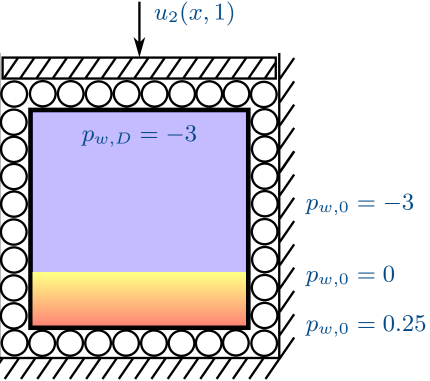
$$\mathbf{x}_{k+1} = \mathbf{g}(\mathbf{x}_k)$$

Idea - Multi-secant quasi-Newton's method:

- Given $m+1$ previous iterates: $\mathbf{x_{k-m}}, ..., \mathbf{x_k}$
- Define $\mathbf{F}_k = \left[\mathbf{g}(\mathbf{x}_{k-m}) - \mathbf{x}_{k-m}, ..., \mathbf{g}(x_k) - \mathbf{x}_k\right]$
- Solve $\alpha = \underset{\sum_i \alpha_i = 1}{\text{argmin}} \|\mathbf{F}_k \alpha\|_2$
- Set $\mathbf{x}_{k+1} = \sum_i \alpha_i \mathbf{g}(\mathbf{x}_{k-m+i})$.

Numerical results: variably saturated porous media

Saturated/unsaturated test case:

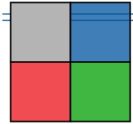
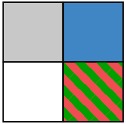


- Domain $\Omega = (0,1)^2$
- Discontinuous initial data
- Volume term for flow problem.
- Hydraulics – Van-Genuchten:

$$
\begin{aligned}
s_w(p_w) &= \begin{cases} (1 + (-\alpha p_w)^n)^{-m} & , p_w \leq 0 \\ 1 & , \text{else} \end{cases} \\
k_{rw}(s_w) &= k\sqrt{s_w}\left(1 - \left(1 - s_w^{1/m}\right)^m\right)^2
\end{aligned}
$$

- $N \times N$ quads, $N \in \{25, 50, 100, 200\}$

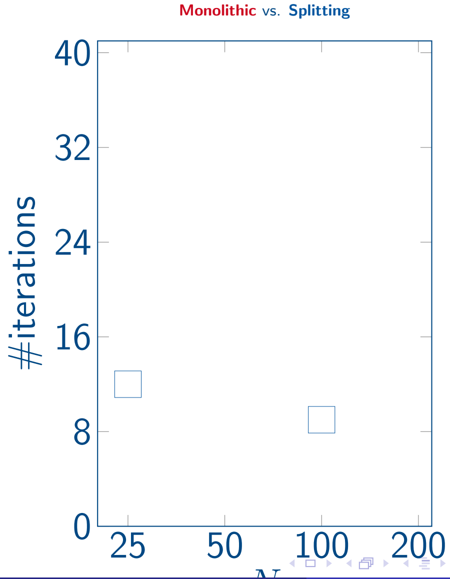
Numerical Results – 8 numerical schemes

	Newton	L-scheme	L-Newton	L-Anderson
				
				

Numerical Results – 8 numerical schemes

Observations:

– **Newton:**
not robust, fastest if successful



Numerical Results – 8 numerical schemes

Observations:

– **Newton:**
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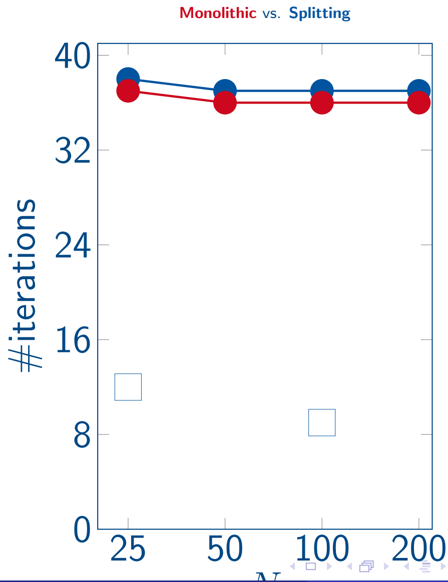
– **L-scheme:** $(L = 0.15)$
robust, slow



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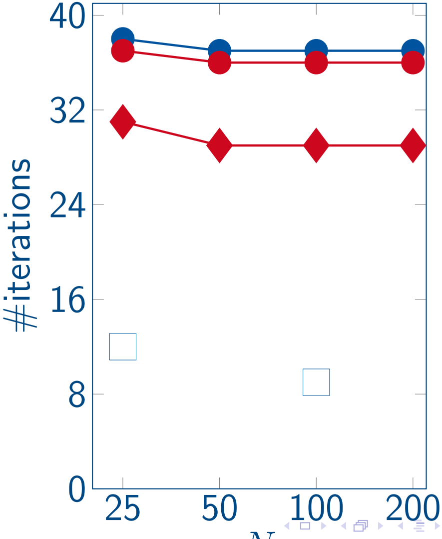
faster than L-scheme
diverging for $m > 1$

Monolithic vs. **Splitting**



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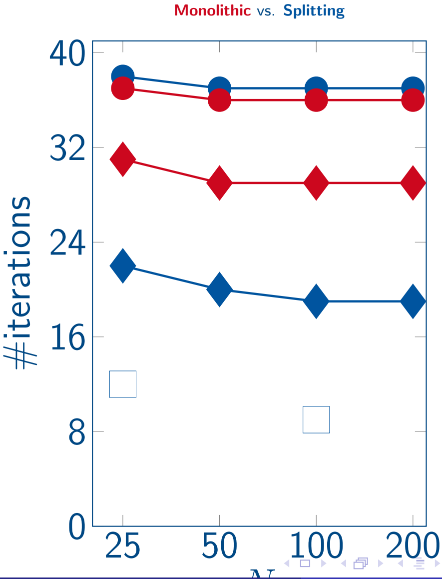
♦ – **L-Anderson:** $(L = 0.02,\ m = 10)$
faster than ♦ due to smoothing



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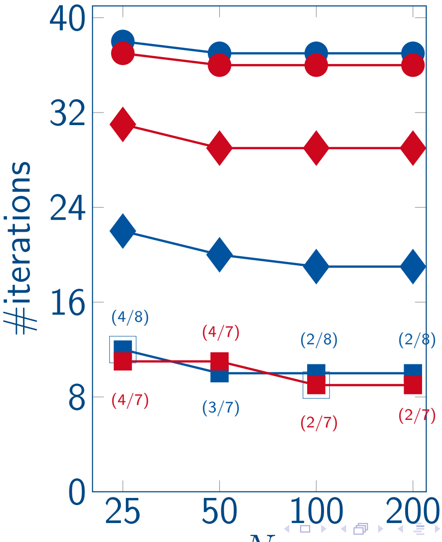
– **L-Anderson:** $(L = 0.02,\ m = 10)$
faster than due to smoothing

– **L-Newton:**
retain Newton's performance

Monolithic vs. **Splitting**



Example 3: A nonlinear Biot model k (div(u))

Mechanics:

$$-\nabla\cdot\Big[\underbrace{\underbrace{2\mu\varepsilon(\mathbf{u})+\lambda\nabla\cdot\mathbf{u}\,\mathbf{I}}_{\text{elastic stress}}-\alpha\underbrace{p}_{\text{pore pressure}}\mathbf{I}}_{\text{poroelastic stress}}\Big] = \mathbf{f}$$

$$\varepsilon(\mathbf{u}) := \frac{\nabla\mathbf{u}+\nabla\mathbf{u}^T}{2}$$

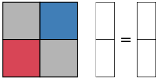
Flow:

$$\underbrace{1/M}_{\text{compressibility}}\partial_t p+\underbrace{\alpha\partial_t\nabla\cdot\mathbf{u}}_{\text{coupling}}-\nabla\cdot\mathbf{k}(\operatorname{div}\mathbf{u})(\nabla\mathbf{p}-\rho\mathbf{g}) = 0$$

$\mathbf{u}$	displacement	$\mu,\ \lambda$	Lamé parameters	ρ	fluid density
p	fluid pressure	α	Biot coefficient	$1/M$	compr. coef.
$\mathbf{q}$	fluid flux	$\mathbf{I}$	identity tensor	k	mobility
$\varepsilon(\mathbf{u})$	linear strain	$\mathbf{f}$	volume force	$\mathbf{g}$	gravity

More details: Kraus, Kumar, Lymbery, R., A fixed-stress splitting method for nonlinear poroelasticity, Engineering with Computers 2024.

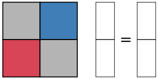
A fixed-stress splitting scheme (after time discretization)



– Monolithic:

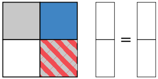
$$\frac{1}{M}p^n + \alpha\nabla\cdot\mathbf{u^n} + \tau\nabla\cdot\mathbf{k}(\mathrm{div}\,\mathbf{u^n})\nabla\mathbf{p^n} = \mathrm{Res}(\mathbf{u^{n-1}},\mathbf{p^{n-1}}),$$

$$-\nabla\cdot\left[\mathbf{2}\mu\varepsilon\left(\mathbf{u^n}\right) + \lambda\nabla\cdot\mathbf{u^n}\mathbf{I} - \alpha\mathbf{p^n}\mathbf{I}\right] = f.$$



– **Monolithic:**

$$
\begin{aligned}
\frac{1}{M}p^n + \alpha\nabla\cdot\mathbf{u^n} + \tau\nabla\cdot\mathbf{k}(\mathrm{div}\,\mathbf{u^n})\nabla\mathbf{p^n} &= \mathrm{Res}(\mathbf{u^{n-1}},\mathbf{p^{n-1}}),\\
-\nabla\cdot\left[\mathbf{2}\mu\varepsilon\left(\mathbf{u^n}\right) + \lambda\nabla\cdot\mathbf{u^n}\mathbf{I} - \alpha\mathbf{p^n}\mathbf{I}\right] &= f.
\end{aligned}
$$



– **Fixed-Stress Splitting Algorithm** (Kraus, Kumar, Lymbery, R., Engineering with Computers 2024.):

$$
\begin{aligned}
\frac{1}{M}p^{n,i+1} + \alpha\nabla\cdot\mathbf{u^{n,i}} + \mathbf{L}(\mathbf{p^{n,i+1}} - \mathbf{p^{n,i}}) - \tau\nabla\cdot\mathbf{k}(\mathrm{div}\,\mathbf{u^{n,i}})\nabla\mathbf{p^{n,i+1}} &= \mathrm{Res}(\mathbf{u^{n-1}},\mathbf{p^{n-1}}),\\
-\nabla\cdot\left[\mathbf{2}\mu\varepsilon\left(\mathbf{u^{n,i+1}}\right) + \lambda\nabla\cdot\mathbf{u^{n,i+1}}\mathbf{I} - \alpha\mathbf{p^{n,i+1}}\mathbf{I}\right] &= f.
\end{aligned}
$$

(A1) We assume that the hydraulic conductivity $K : \mathbb{R} \mapsto \mathbb{R}$ is differentiable, strictly positive and Lipschitz continuous with a Lipschitz constant K_L, i.e., the function K satisfies the conditions

$$K \in C^1(\mathbb{R}), \tag{18a}$$

$$0 < \underline{K} \leq K(z) \text{ for all } z \in \mathbb{R}, \tag{18b}$$

$$|K(z_1) - K(z_2)| \leq K_L|z_1 - z_2| \text{ for all } z_1, z_2 \in \mathbb{R}. \tag{18c}$$

(A2) We assume that there holds $\nabla p^n \in L^\infty(\Omega)$.

Theorem (Kraus, Kumar, Lymbery, R., Engineering with Computers 2024)
Assuming (A1) and (A2), the fixed-stress scheme converges for a sufficiently small time step τ and for $L \geq \dfrac{1}{d^{-1}+\lambda}$, where d is the spatial dimension. In particular, the following estimates holds:

$$\|e_p^{i+1}\| \leq \sqrt{\frac{c_0 + \frac{\tau}{4K}\frac{c^2}{(c_K^2+\lambda)^2}}{c_1}}\|e_p^i\|,$$

$$\left(\|\epsilon(\mathbf{e_u^{i+1}})\|^2 + \lambda\|\mathrm{div}\,\mathbf{e_u^{i+1}}\ \|^2\right)^{\frac{1}{2}} \leq \sqrt{\frac{1}{c_K^2+\lambda}}\|e_p^i\|,$$

where $c_0 := L/2 < c_1 := L/2 + (\beta_s^{-2}+\lambda)^{-1}/2$, $c_K = 1/\sqrt{d}$ and β_s the Stokes inf-sup constant, i.e.

$$\inf_{q\in L_0^2(\Omega)} \sup_{\mathbf{w}\in\mathbf{H_0^1(\Omega)}} \frac{(\mathrm{div}\,\mathbf{w}, q)}{\|\mathbf{w}\|_1\|q\|} \geq \beta_s.$$

A numerical example

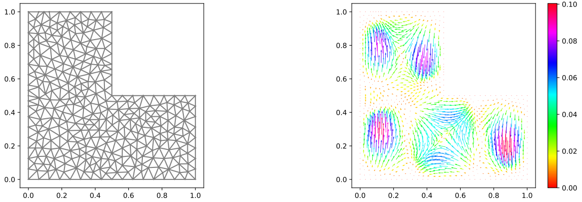


Figure: Left: triangulation of Ω, $h = 1/16$; right: $\mathbf{u}$, where $h = 1/32$, $\tau = 0.01$, $S = 10^{-4}$, $K_0 = 10^{-6}$, and $\lambda = 10^2$.

Number of iterations for the linear model

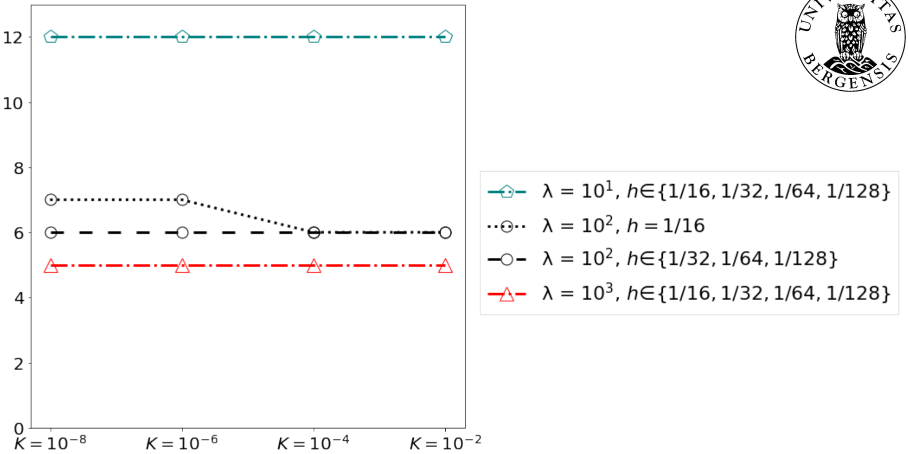


Figure: Number of iterations for the linear model for fixed time step size $\tau = 0.01$, $M = 10^4$, varying the mesh size h, λ, and K.

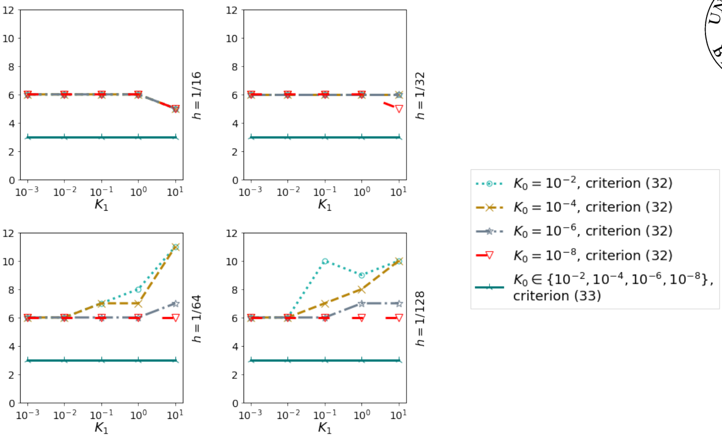


Figure: Number of iterations for $K = K(\operatorname{div} \mathbf{u}) = K_0 + K_1(\operatorname{div} \mathbf{u})^2$. Time step size $\tau = 0.01$, $M = 10^4$, and $\lambda = 10^2$.

Different nonlinear Biot models: summary

• Nonlinear parameters [Borregales, R., Kumar, Nordbotten, Comp. Geo. 2018] Assuming that the nonlinearities are Lipschitz continuous and monotone increasing, and that the stabilization parameters are big enough, then all the considered schemes are at least linearly convergent. The only quadratic convergent scheme is the monolithic Newton.

• Large deformations case [Borregales, Kumar, Nordbotten, R., Comp. Geo.i 2021] Difficult to prove something theoretically. Many different schemes were (numerically) tested. Convincing numerical results.

• Saturated/unsaturated flow in deformable porous media [Both, Kumar, Nordbotten, R., CAMWA 2019]. Anderson acceleration (stabilizing effect!).

• Nonlinear permeability [Kraus, Kumar, Lymbery, R., Engineering with Computers 2024Both, Kumar, Nordbotten, R., CAMWA 2019].

The abstract problem

Let V_1, V_2, V_3 be Hilbert (Banach) spaces. Let $a_j(\cdot,\cdot) : V_j \times V_j \to \mathbb{R}$, $j = 1,2,3$, $b_{ij}(\cdot,\cdot) : V_j \times V_i \to \mathbb{R}$, $i,j = 1,2,3$ be bilinear, $F_j : V_j \to \mathbb{R}$, $j = 1,2,3$ be linear. The variational problem we want to solve reads:

Find $(u,p,T) \in V_1 \times V_2 \times V_3$ s.t. there holds for all $(v,w,Q) \in V_1 \times V_2 \times V_3$

$$\left\{ \begin{array}{rcl} a_1(u,v) - b_{12}(p,v) - b_{13}(T,v) & = & F_1(v), \\ a_2(p,w) + b_{21}(u,w) - b_{23}(T,w) & = & F_2(w), \\ a_3(T,Q) + b_{31}(u,Q) + b_{32}(p,Q) & = & F_3(Q). \end{array} \right. \tag{19}$$

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Example thermo-poromechanics: See please Brun, Ahmed, Berre, Nordbotten, Radu, Monolithic and splitting solution schemes for thermo-poroelasticity with nonlinear convective transport, CAMWA 2020.

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Plan:

- Set up a robust splitting scheme.
- Analysis: convergence.
- Example: fixed-stress splitting for thermo-poromechanics.
- Generalizations: n-equations, nonlinear case.

A robust splitting scheme

Let $(u^0, p^0, T^0) \in V_1 \times V_2 \times V_3$ be given. For $i \geq 0$, find $(u^{i+1}, p^{i+1}, T^{i+1}) \in V_1 \times V_2 \times V_3$ s.t. there holds for all $(v, w, Q) \in V_1 \times V_2 \times V_3$

$$\left\{ \begin{array}{rcl} a_1(u^{i+1}, v) - b_{12}(p^{i+1}, v) - b_{13}(T^{i+1}, v) & = & F_1(v), \\ a_2(p^{i+1}, w) + b_{21}(u^i, w) - b_{23}(T^{i+1}, w) + L_2\langle p^{i+1} - p^i, w\rangle & = & F_2(w), \\ a_3(T^{i+1}, Q) + b_{31}(u^i, Q) + b_{32}(p^i, Q) + L_3\langle T^{i+1} - T^i, Q\rangle & = & F_3(Q). \end{array} \right. \tag{20}$$

Notation: i the iteration index.

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- $L_2, L_3 \geq 0$ are (free to chosen) stabilization parameters.
- Consistency.
- Anderson acceleration.
- Convergence: To show that $e_p^i \to 0, e_u^i \to 0, e_T^i \to 0$ where $e_p^i := p - p^i, e_u^i := u - u^i, e_T^i := T - T^i$.
- Stabilization, optimization.
- Many different splitting schemes are possible!

Notation: i the iteration index.

Analysis

Assumptions

- $a_j(\cdot,\cdot)$, $j = 1, 2, 3$ are bilinear, coercive $(a_{j,\star})$ and continuous (M_{a_j}).
- $b_{ij}(\cdot,\cdot)$ are bilinear and continuous $(M_{b_{ij}})$, and $b_{ij} = b^t_{ji}$, $i, j = 1, 2, 3$.
- $F_j(\cdot)$, $j = 1, 2, 3$ are linear and continuous.

Theorem

The iterative scheme (20) *converges for*

$$\begin{aligned} L_2 &\geq \frac{M^2_{b_{23}}}{a_{3,\star}} + \frac{M^2_{b_{12}}}{a^2_{1,\star}} \left(\frac{M^2_{b_{31}}}{a_{3,\star}} + \frac{M^2_{b_{12}}}{a_{2,\star}} \right), \\ L_3 &\geq \frac{M^2_{b_{13}}}{a^2_{1,\star}} \left(\frac{M^2_{b_{31}}}{a_{3,\star}} + \frac{M^2_{b_{12}}}{a_{2,\star}} \right). \end{aligned}$$

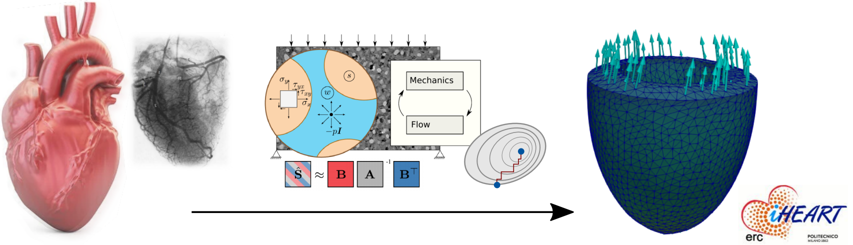
Notation: i the iteration index, $e^i_p := p - p^i, e^i_u := u - u^i, e^i_T := T - T^i$.

Analysis: remarks

- One can use the analysis to choose the splitting scheme, depending on the values of coercivity constants.
- Some assumptions can be relaxed to include degenerate cases.
- Similar schemes and analysis can be done for the nonlinear version of (19). Obs.: a_j nonlinear: easy; b_{ij} nonlinear: difficult!
- Example 1: Both, Barnafi, Radu, Zunino, Quarteroni, Iterative splitting schemes for a soft material poromechanics model, CMAME 2022.
- Example 2: thermo-poromechanics, e.g. in Brun, Ahmed, Berre, Nordbotten, Radu, Monolithic and splitting solution schemes for thermo-poroelasticity with nonlinear convective transport, CAMWA 2020.

Soft tissue poromechanics

- Mathematical model by Chapelle, Moireau, Eur J Mech B-Fluid 2014
- Dynamic (includes acceleration terms), nonlinear model
- Applications: perfusion of the heart
- Details (linearization, solvers, analysis): Both, Barnafi, R., Zunino, Quarteroni, CMAME 2022.



Linearized soft tissue poromechanics model

Continuous formulation:

$$\rho_s(1-\phi)\partial_{tt}\boldsymbol{u} - \nabla\cdot\mathbb{C}\varepsilon(\mathbf{u}) + (1-\phi)\nabla\mathbf{p} - \phi^2\boldsymbol{\kappa}^{-1}(\boldsymbol{v}_f - \partial_t\boldsymbol{u}) = \rho_s(1-\phi)\boldsymbol{f},$$

bulk-fluid momentum balance

$$\rho_f\phi\partial_t\boldsymbol{v}_f - \nabla\cdot(\phi\mathbf{2}\mu_f\varepsilon(\mathbf{v_f})) + \phi\nabla\mathbf{p} + \phi^2\boldsymbol{\kappa}^{-1}(\boldsymbol{v}_f - \partial_t\boldsymbol{u}) = \rho_f\phi\mathbf{f},$$

fluid momentum balance

$$\frac{\rho_f(1-\phi)^2}{\kappa_s}\partial_t p + \nabla\cdot(\rho_f\phi\mathbf{v_f}) + \nabla\cdot(\rho_f(\mathbf{1}-\phi)\partial_\mathbf{t}\mathbf{u}) = \theta.$$

mass conservation

[2] Barnafi et al., (2021)

Linearized soft tissue poromechanics model

Continuous formulation:

$$\rho_s(1-\phi)\partial_{tt}\boldsymbol{u} - \nabla\cdot\mathbb{C}\varepsilon(\mathbf{u}) + (1-\phi)\nabla\mathbf{p} - \phi^2\boldsymbol{\kappa}^{-1}(\boldsymbol{v}_f - \partial_t\boldsymbol{u}) = \rho_s(1-\phi)\boldsymbol{f},$$

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fluid momentum balance

$$\frac{\rho_f(1-\phi)^2}{\kappa_s}\partial_t p + \nabla\cdot(\rho_f\phi\mathbf{v_f}) + \nabla\cdot(\rho_f(\mathbf{1}-\phi)\partial_\mathbf{t}\mathbf{u}) = \theta.$$

mass conservation

Inf-sup CG-FEM[2] + IE discretization:

$$\begin{bmatrix} \mathbf{A}+\mathbf{M} & & -\mathbf{B}_1^\top \\ & & -\mathbf{B}_2^\top \\ \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{M}_p \end{bmatrix} \begin{bmatrix} \mathbf{u}^n \\ \mathbf{v}^n \\ \mathbf{p}^n \end{bmatrix} = \begin{bmatrix} -\mathbf{R}_\mathbf{u}^n \\ -\mathbf{R}_\mathbf{v}^n \\ -\mathbf{R}_\mathbf{p}^n \end{bmatrix}$$

[2] Barnafi et al., (2021)

Stabilization suggested by convergence theory[5]

General stabilization ansatz:

$$
\left(\underbrace{\begin{bmatrix} \mathbf{L}_{\mathrm{s}} & & \\ & \mathbf{L}_{\mathrm{f}} & \\ & & \mathbf{L}_{\mathrm{p}} \end{bmatrix}}_{\text{(Stabilization)}} + \underbrace{\begin{bmatrix} \mathbf{A}_{\mathrm{s}} & \mathbf{A}_{\mathrm{sf}} & -\mathbf{B}_1^\top \\ & \mathbf{A}_{\mathrm{f}} & -\mathbf{B}_2^\top \\ & \mathbf{B}_2 & \mathbf{M}_{\mathrm{p}} \end{bmatrix}}_{\text{(Decoupled } \mathbf{u}\text{)}} \right) \begin{bmatrix} \Delta \mathbf{u}^{n,k} \\ \Delta \mathbf{v}^{n,k} \\ \Delta \mathbf{p}^{n,k} \end{bmatrix} = \begin{bmatrix} -\mathbf{R}_{\mathbf{u}}^{n,k} \\ -\mathbf{R}_{\mathbf{v}}^{n,k} \\ -\mathbf{R}_{\mathbf{p}}^{n,k} \end{bmatrix}
$$

Lemma (Abstract convergence result)

Let:

$$\{x_k\}_k \subset \mathbb{R}_+$$

$$\sum_{i=1}^{\infty} x_{k+i} \leq C x_k \ \forall k$$

$\Rightarrow x_{k_l} \leq \mathrm{rate}(C)^{k_l} x_0$ *(r-linear conv.)*

Theorem (Convergence of diagonal stabilization)

r-linear convergence if:

$$\mathbf{L}_{\mathrm{s}} \succeq -\frac{\phi^2 \kappa^1}{2\Delta t} \mathbf{M}_{L^2} = -\tfrac{1}{2}\mathbf{K}_{\mathrm{s}} \quad \textit{(destabilization)}$$

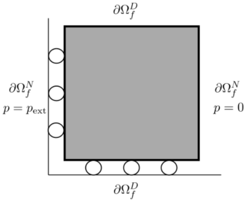
$$\mathbf{L}_{\mathrm{f}} \succeq \mathbf{0}$$

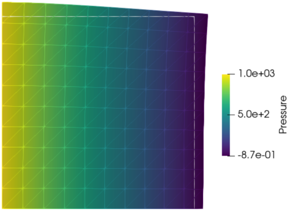
$$\mathbf{L}_{\mathrm{p}} \succeq \frac{(1-\phi)^2}{K_{\mathrm{dr}}} \mathbf{M}_{L^2} \quad \textit{(fixed-stress)}$$

[5] Both, Barnafi, Radu, Zunino, Quarteroni (CMAME 2022) Iterative splitting schemes for a soft material poromechanics model

Swelling test

Setup:





Fixed parameters: $\rho_{\mathrm{f}} = \rho_{\mathrm{s}} = 1000$, $\mu_{\mathrm{f}} = 0.035$, $\lambda_{\mathrm{s}} = 711$, $\mu_{\mathrm{s}} = 4066$, $\kappa_{\mathrm{s}} = 10^3$, $\kappa_{\mathrm{f}} = 10^{-7}\mathbf{I}$, $\phi = 0.1$

Sensitivity wrt. compressibility:

κ_{s}	# avg. iters.
10^2	8.55
10^3	15.91
10^4	64.09
10^5	–

Sensitivity wrt. permeability:

κ_{f}	# avg. iters.
10^{-9}	17.64
10^{-10}	73.72
10^{-11}	399.96
10^{-12}	–

Notation – L^2-stabilization

ID	$\bar{\beta}_{\mathrm{s}}$	$\bar{\beta}_{\mathrm{f}}$	$\bar{\beta}_{\mathrm{p}}$	Description	Theory
$L^2S_{0,0,0}$	0	0	0	Unstabilized split	✗
$L^2S_{0,0,1}$	0	0	1	L^2S with fixed-stress-type p-stabilization	✓
$L^2S_{-0.5,0,1}$	$-\frac{1}{2}$	0	1	L^2S with conservative $\boldsymbol{u}$-destabilization	✓
$L^2S_{-1,0,1}$	-1	0	1	L^2S with aggressive $\boldsymbol{u}$-destabilization	✗

Table: Considered stabilization settings in the context of the diagonally L^2–stabilized split.

Swelling test – Sensitivity of L^2 stab wrt. compressibility

Fixed-stress-like split:

κ_s	P1/P1/P1 elements $L^2S_{0,0,0}$	$L^2S_{0,0,1}$	$L^2S_{-0.5,0,1}$	$L^2S_{-1,0,1}$	P1/P2/P1 elements $L^2S_{0,0,0}$	$L^2S_{0,0,1}$	$L^2S_{-0.5,0,1}$	$L^2S_{-1,0,1}$
10^2	6.73	6.0	5.91	5.82	6.73	6.73	6.36	6.36
10^4	13.27	22.81	22.82	22.91	13.18	7.0	6.73	6.91
10^6	–	–	–	–	14.0	7.09	6.82	7.0
10^8	–	–	–	–	14.09	7.09	6.82	7.0

Swelling test – Sensitivity wrt. permeability

Fixed-stress-like split (AA(0)):

κ_f	P1/P1/P1 elements				P1/P2/P1 elements			
	$L^2S_{0,0,0}$	$L^2S_{0,0,1}$	$L^2S_{-0.5,0,1}$	$L^2S_{-1,0,1}$	$L^2S_{0,0,0}$	$L^2S_{0,0,1}$	$L^2S_{-0.5,0,1}$	$L^2S_{-1,0,1}$
10^{-7}	10	8.18	8.18	8.36	10	6.36	6.27	6.64
10^{-8}	12	9.91	9.82	9.91	11.91	9	8.45	8
10^{-9}	15.09	15.09	12.36	11.18	15.45	15.36	12.55	9.55
10^{-10}	67.18	67.28	40	55	74.64	74.73	44.27	19.27
10^{-11}	347.55	348.18	194	–	419.64	420.45	232	–
10^{-12}	–	–	–	–	–	–	–	–

Swelling test – Sensitivity wrt. permeability

Fixed-stress-like split (AA(0)):

κ_f	P1/P1/P1 elements $L^2S_{0,0,0}$	$L^2S_{0,0,1}$	$L^2S_{-0.5,0,1}$	$L^2S_{-1,0,1}$	P1/P2/P1 elements $L^2S_{0,0,0}$	$L^2S_{0,0,1}$	$L^2S_{-0.5,0,1}$	$L^2S_{-1,0,1}$
10^{-7}	10	8.18	8.18	8.36	10	6.36	6.27	6.64
10^{-8}	12	9.91	9.82	9.91	11.91	9	8.45	8
10^{-9}	15.09	15.09	12.36	11.18	15.45	15.36	12.55	9.55
10^{-10}	67.18	67.28	40	55	74.64	74.73	44.27	19.27
10^{-11}	347.55	348.18	194	–	419.64	420.45	232	–
10^{-12}	–	–	–	–	–	–	–	–

Fixed-stress-like split (AA(5)):

κ_f	P1/P1/P1 elements $L^2S_{0,0,0}$	$L^2S_{0,0,1}$	$L^2S_{-0.5,0,1}$	$L^2S_{-1,0,1}$	P1/P2/P1 elements $L^2S_{0,0,0}$	$L^2S_{0,0,1}$	$L^2S_{-0.5,0,1}$	$L^2S_{-1,0,1}$
10^{-7}	5.9	5.73	6	6	5.73	4.91	4.91	4.91
10^{-8}	7	7.27	7.27	7.09	6.91	6.91	6.64	5.91
10^{-9}	10.36	10	8.91	8.91	10.45	10	9	7.09
10^{-10}	18.91	18.09	14.91	12	18	20.09	15.73	10
10^{-11}	43.55	45.18	33.73	26.18	56.82	53.18	38.91	18.82
10^{-12}	107.09	112.73	121.55	–	140.73	117.36	95.64	280.82

Summary and Outlook

- We reviewed some convergence results for solving two or more coupled (linear or nonlinear) PDEs, with a special focus on fixed-stress scheme for the quasi-static, linear Biot model.
- We show how one can optimize the fixed-stress splitting scheme.
- One can obtain splitting schemes by using the Schur complement and a matrix decomposition.
- Applications in poromechanics, surfactant transport, phase-fields, hydrology (GRW), contact mechanics....
- Extend the schemes to even more complex models (multiphase flow, reactive flow)
- Look at different dynamic Biot model (seismicity)

Thank you for your attention!

Acknowledgments: VISTA CSD

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