Multiwavelet troubled-cell indicator for discontinuity detection of discontinuous Galerkin schemes

Thea Vuik
Delft University of Technology

Collaboration with Jennifer Ryan, University of East Anglia

May 23, 2014
Motivation

Flow around Space Shuttle

Solution linear advection equation
Outline

1. Discontinuous Galerkin
2. Limiters and troubled-cell indicators
3. Multiwavelets
4. Multiwavelet troubled-cell indicator
5. Numerical examples (1d Euler equations)
6. Numerical example (2d Euler equations)
7. Conclusion
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Discontinuous Galerkin

Hyperbolic partial differential equation:
\[ u_t + f(u)_x = 0; \quad x \in [-1, 1], \quad t \geq 0. \]

- DG approximation: for \( x \in I_j \), write,
  \[ u_h(x) = \sum_{\ell=0}^{k} u_j^{(\ell)} \phi_{\ell}(\xi_j), \quad \xi_j = \frac{2}{\Delta x} (x - x_j) \]

- approximation space: orthonormal Legendre polynomials
  \[ \int_{-1}^{1} \phi_{\ell}(x) \phi_{m}(x) dx = \delta_{\ell m} \]

- \( k \): highest polynomial degree of the approximation
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Limiters

Limiter:

- Helps to control spurious oscillations
- Reduces polynomial order in nonsmooth regions
- May flatten local extrema (diffusive property)

Troubled-cell indicator:

- Helps to limit at discontinuities only
Troubled-cell indicators

Examples of troubled-cell indicators for DG:

- minmod-based TVB limiter
- KXRCF indicator
- Harten’s subcell resolution

These indicators use local information (neighbouring cells)
Multiwavelet approach: global and local information
Multiwavelets


- specific set of piecewise polynomials
- based on orthonormal Legendre polynomials
- possible to decompose function into several levels

Basis spans piecewise polynomials on $[-1, 0] \cup [0, 1]$, degree $\leq 2$
Multiwavelet decomposition: next levels

Relation between DG and multiwavelets ($2^n$ elements):

\[ u_h(x) = \sum_{j=0}^{2^n-1} \sum_{\ell=0}^{k} u_j^{(\ell)} \phi_\ell(\xi_j) = S^0(x) + \sum_{m=0}^{n-1} D^m(x) \]
Multiwavelet decomposition

Example uses $n = 2$: 4 elements on $[-1, 1]$

$$u_h(x) = \sum_{j=0}^{3} \sum_{\ell=0}^{k} u_j^{(\ell)} \phi_\ell(\xi_j) = S^0(x) + \sum_{m=0}^{n-1} D^m(x), \quad n - 1 = 1$$

<table>
<thead>
<tr>
<th>$S^0(x)$</th>
<th>average</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D^0(x)$</td>
<td>coarse details</td>
</tr>
<tr>
<td>$D^1(x)$</td>
<td>finer details</td>
</tr>
<tr>
<td>$l_0$</td>
<td>$l_1$</td>
</tr>
</tbody>
</table>

Regions where multiwavelet contributions are continuous

Both $D^1(x)$ and $u_h(x)$: continuous on $l_0, \ldots, l_3$
Highest level

- $D^{n-1}$ constructed using $d^{n-1} = (d^{n-1}_0 \ldots d^{n-1}_k)^\top$
- Jump between cells: $\Delta u = ([u_h]^{(0)} \ldots [u_h]^{(k)})^\top$

\[
\begin{array}{|c|c|}
\hline
\text{Left} & \text{Right} \\
\hline
\end{array}
\]

\[
d^{n-1}, \Delta u
\]

\[
d^{n-1} = A\Delta u,
\]

where

\[
A(\ell + 1, r + 1) = 2^{-\frac{n-1}{2}} \frac{2^{(-n+1)r}}{r!} \int_0^1 x^r \psi_\ell(x) \, dx.
\]
Highest level

This means that $D^{n-1}$:

- Measures jumps in approximation (derivatives) at element boundaries;
- Can be used for detection of discontinuities (in derivatives).
Continuous example

Most details are visible in $D^{n-1}(x)$

Example: use $n = 6$: $2^6$ elements

Multiwavelet approximation $D^5(x)$ of $\sin(2\pi x)$
Discontinuous example

Most details are visible in $D^{n-1}(x)$

Example: use $n = 6$: $2^6$ elements

Multiwavelet approximation $D^5(x)$ of square wave
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Multiwavelet troubled-cell indicator

- Troubled cells: focus on highest level $D^{n-1}(x)$
- Compute absolute average $\bar{D}_j^{n-1}$ on element $I_j$
- Element $I_j$ is troubled cell if,

$$\bar{D}_j^{n-1} \geq C \cdot \max \left\{ \bar{D}_i^{n-1}, i = 0, \ldots, 2^n - 1 \right\}, \quad C \in [0, 1]$$
Choice of $C$

$l_j$ is troubled cell if,

$$\bar{D}_j^{n-1} \geq C \cdot \max \left\{ \bar{D}_i^{n-1}, \ i = 0, \ldots, 2^n - 1 \right\}, \ C \in [0,1]$$

Parameter $C$: defines strictness of indicator,

- $C = 0$: every element is detected
- $C = 0.2$: select largest 80% of averages
- $C = 0.8$: select largest 20% of averages
Multiwavelet troubled-cell indicator

Applications: Euler equations

- Local detector: shock in different locations
  (Zaide and Roe, 20th AIAA CFD Conf. 2011)

  Our indicator: combines local and global nature

- Limiter: mechanism to control limited regions
  Now: troubled-cell indicator as switch


  Only a choice, other limiters possible
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Density in Sod’s shock tube at $T = 0$ (left) and $T = 2$ (right)
Sod: time history

Results: focus on detected troubled cells

Time history of troubled cells
Approximation, $t = 0.017375$
Sine entropy wave:

\[ \rho(x, 0) = \begin{cases} 
3.857142, & x < -4, \\
1 + 0.2 \sin(5x), & x \geq -4.
\end{cases} \]

Two-dimensional approach

In two-dimensions, the multiwavelet expansion is:

\[ S^0(x, y) + \sum_{m_x=0}^{n_x-1} \sum_{m_y=0}^{n_y-1} \left\{ D^{\alpha,m}(x, y) + D^{\beta,m}(x, y) + D^{\gamma,m}(x, y) \right\} \]

number of elements: \(2^{n_x} \times 2^{n_y}\)

- \(\alpha\) mode: multiwavelets in \(y\)-direction
- \(\beta\) mode: multiwavelets in \(x\)-direction
- \(\gamma\) mode: multiwavelets both \(x\)- and \(y\)-direction
Double Mach reflection

Density contours using $C = 0.05$

$T = 0.2$, $\Delta x = \Delta y = \frac{1}{128}$, $k = 1$

Detected troubled cells

Detected troubled cells at $T = 0.2, \ C = 0.05$

Different troubled cells are detected by modes
Computation time

Compare computation time, double Mach reflection:

- More accurate result: don’t limit continuous regions
- Decrease of computation time

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>limit everywhere</th>
<th>$C = 0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$512 \times 128$</td>
<td>57</td>
<td>50</td>
</tr>
<tr>
<td>$1024 \times 256$</td>
<td>493</td>
<td>441</td>
</tr>
</tbody>
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Computation time in minutes, $T = 0.2$, $k = 1$
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Conclusion

Troubled-cell indicator is switch in limiter
Multiwavelet decomposition: $D^{n-1}$ detects discontinuity
Parameter C defines strictness of detector
More accurate than existing detectors
Two-dimensional detection in different modes
Decrease of computation time

More details in JCP(270), pp 138-160

Future work:
→ How to choose parameter C
→ Applying to unstructured meshes