Low-dose discrete tomography: algorithm and Matlab implementation

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Tomography
parallel beams

object $\mathcal{X}$

sources

FOV

detectors

$\theta$

$p$
Linear model

Based on line integrals:

$$\int_L f(\xi) \, d\xi \approx \sum_{j=1}^N W_{ij} x_j$$

The set of projections result in a linear system of equations

$$Wx = p$$
Photon limited data leads to ill-posed problems

Low exposure times

\[ Wx = p + \epsilon \]

Limited view angles

\[ W \in \mathbb{R}^{M \times N} \]

\[ M \ll N \]
Ill-posed problems

Regularization through prior knowledge...
Ill-posed problems

Regularization through prior knowledge...
Reconstruct an image with few grey values, known a priori. In this example two values $(\rho_1, \rho_2)$:

$$\min_{x \in \{\rho_1, \rho_2\}^N} \| Wx - p \|_2$$
Enforcing discrete solutions

segmentation

ground truth | LSQR | segmented
Enforcing discrete solutions

segmentation

ground truth  LSQR  error
Refining the boundary

Interior pixels are removed from the equation system

\[
\begin{pmatrix}
  w_1 & \cdots & w_{i-1} & w_{i+1} & \cdots & w_N
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_{i-1} \\
  x_i \\
  \vdots \\
  x_N
\end{pmatrix}
= p - v_i w_i,
\]

and the boundary is updated by solving the reduced system...
Refining the boundary

Interior pixels are removed from the equation system

\[
\begin{pmatrix}
  w_1 & \cdots & w_{i-1} & w_{i+1} & \cdots & w_N
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= p - v_i w_i,
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and the boundary is updated by solving the reduced system...
Discrete Algebraic Reconstruction Technique (DART)

DART reconstruction from 15 projections

ground truth  segmentation  error
Discrete Algebraic Reconstruction Technique (DART)

DART reconstruction from 15 projections

- Ground truth
- Segmentation
- Error
DART on noisy projections

Noise accumulates in the boundaries, due to DART’s hard constraints...
Update after computing the segmentation:

**DART:**

\[
\min_{x \in \mathbb{R}^N} \| Wx - p \|_2^2 \quad \text{s.t.} \quad x_i = v_i, \text{ for } i \notin B \quad \text{(Boundary)}
\]

**SDART:**

\[
\min_{x \in \mathbb{R}^N} \| Wx - p \|_2^2 + \lambda^2 \| D(x - v) \|_2^2
\]
Soft DART

Pixels are weighted based on the pixel values of their neighbors.

\[ D_{ii} := \frac{1}{\alpha^b_i} \]

\( b_i \) - number of neighbors with a different greyvalue
Soft DART

Results

ground truth  SIRT  DART  SDART

photon count  pixel error percentage

DART  SDART  SIRT

\[
\text{pixel error percentage} = 10 \times \frac{\text{photon count}}{\text{ground truth}}
\]
Soft DART

Results

ground truth

SIRT

DART

SDART

pixel error percentage vs. photon count

DART
SDART
SIRT
Main step is the boundary refinement...

$$\text{minimize } \min_{x \in \mathbb{R}^N} \left\| \begin{pmatrix} W \\ \lambda D \end{pmatrix} x - \begin{pmatrix} p \\ Dv \end{pmatrix} \right\|_2$$

this matrix is too large to fit in memory!

For a $1k \times 1k$ reconstruction from 100 projections, $W$ is a $100k \times 1M$ matrix
Main step is the boundary refinement . . .

\[
\minimize_{x \in \mathbb{R}^N} \left\| \begin{pmatrix} W \\ \lambda D \end{pmatrix} x - \begin{pmatrix} p \\ Dv \end{pmatrix} \right\|_2
\]

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For a 1k × 1k reconstruction from 100 projections, \( W \) is a 100k × 1M matrix
Main step is the boundary refinement…

\[
\text{minimize}_{x \in \mathbb{R}^N} \left\| \begin{pmatrix} W \\ \lambda D \end{pmatrix} x - \begin{pmatrix} p \\ Dv \end{pmatrix} \right\|_2
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For a $1k \times 1k$ reconstruction from 100 projections, $W$ is a $100k \times 1M$ matrix
ASTRA toolbox
for tomographic reconstruction

Primitives:
- GPU accelerated forward and back projection

Reconstruction algorithms:
- FBP, FDK reconstruction
- Iterative SIRT, CGLS reconstruction

Geometries:
- 2D parallel and fan beam
- 3D parallel and cone beam
- All with fully flexible source/detector positioning
Spot toolbox
for linear operators

Spot toolbox, by Ewout van den Berg and Michael P. Friedlander,
UBC Canada
opTomo
A Spot operator for ASTRA

```matlab
im = phantom(512);
% Spot operator
W = opTomo('cuda', proj_geom, vol_geom)
```

```matlab
p = W*im(:);
% is equivalent to:
sino = astra_create_sino_cuda(im, proj_geom, ...
        vol_geom);
```

```matlab
x = W'*p;
% is equivalent to:
volume = astra_create_backprojection_cuda(p, ...
        proj_geom, vol_geom);
```
while (~converged)

% segmentation
recon_seg = astra.utils.segment(recon, ... geyValues);

% Identify neighbor boundary pixels
nnb = astra.utils.boundary(recon);

D = lambda * diag(sparse((1./3.^((nnb(:))))));

v = recon_seg;

% update step
recon = cglsh([W; D], [p; D*v]);
end
Conclusions

Prior knowledge is essential for reconstructing from limited projection data:

- Classic methods need many angles
- Algebraic reconstruction methods benefit from using prior knowledge

SDART is proposed to improve DART in case of noisy projection data:

- Soft constraints reduce the effect of noise in SDART
- Resulting in substantially improved reconstructions

The Spot toolbox links the high performance ASTRA code with a linear algebra interface