Integer programs with bounded subdeterminants and two nonzeros per row (or column)

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Outline

1. Main results and motivation
2. Reduction to stable set in graphs with bounded OCP
3. Structure of graphs with bounded OCP
4. Main algorithm

Outline

Definition

For $\Delta \in \mathbb{Z}_{\geqslant 0}$, a matrix A is called *totally* Δ*-modular* if

$$\det(A') \in \{-\Delta, -\Delta + 1, \ldots, 0, \ldots, \Delta - 1, \Delta\}$$

for all square submatrices A' of A

Given A, let $\quad \Delta(A) := \min\{\Delta : A \text{ is totally } \Delta\text{-modular}\}$

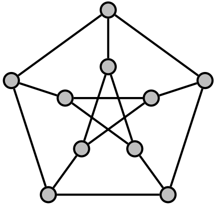
Definition

The *odd cycle packing number* $\operatorname{ocp}(G)$ is the maximum number of vertex-disjoint odd cycles in G

Examples:

- A is totally unimodular (TU) $\iff \Delta(A) \leqslant 1$

- A is the incidence matrix of graph $G \implies \Delta(A) = 2^{\mathrm{ocp}(G)}$



$$\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\end{pmatrix}$$

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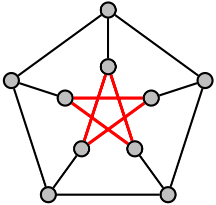

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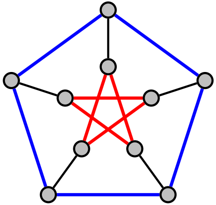

$\mathrm{ocp}(G) = 2$

$$\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$\Delta(A) = 4 = 2^2$

Our main results

Theorem (FJWY '21)

For every integer $\Delta \geq 1$ there exists a strongly polynomial-time algorithm for solving the integer program (IP)

$$\max\{w^\intercal x : Ax \leqslant b,\ x \in \mathbb{Z}^n\}$$

where $w \in \mathbb{Z}^n$, $b \in \mathbb{Z}^m$, and constraint matrix $A \in \mathbb{Z}^{m \times n}$

- *is totally Δ-modular, and*
- *contains at most two nonzero entries in each row (or in each column)*

Theorem (FJWY '21)

For every integer $k \geq 0$ there exists a strongly polynomial-time algorithm for the weighted stable set problem in graphs with $\mathrm{ocp}(G) \leqslant k$

Previous work

• Bock et al' : PTAS for MWSS in graphs with OCP(G) = O(1)

1. (IP) can be solved in strongly polynomial-time if $\Delta = 1$
2. (IP) can be solved in strongly polynomial-time if $\Delta = 2$ (Artmann, Weismantel, Zenklusen **'17**)
3. There is a polynomial-time algorithm that solves (IP) w.h.p. over the choices of b, when A, w are fixed and Δ is constant (Paat, Schlöter, Weismantel **'19**)
4. The diameter of $P := \{x : Ax \leqslant b\}$ is $O(\Delta^2 n^4 \lg n\Delta)$ (Bonifas, Di Summa, Eisenbrand, Hähnle, Niemeier **'14**)
5. $\max\{w^\intercal x : Ax = b,\ x \geqslant 0\}$ can be solved in time $\mathrm{poly}(m, n, \lg \Delta)$ (Tardos **'86**)
6. $\max\{w^\intercal x : Ax = b,\ x \geqslant 0\}$ can be solved in time $O(mn^\omega \lg(n) \lg(\bar{\chi} + n))$ time (Dadush, Natura, Végh **'20**)

Outline

Proximity result of Cook et al.

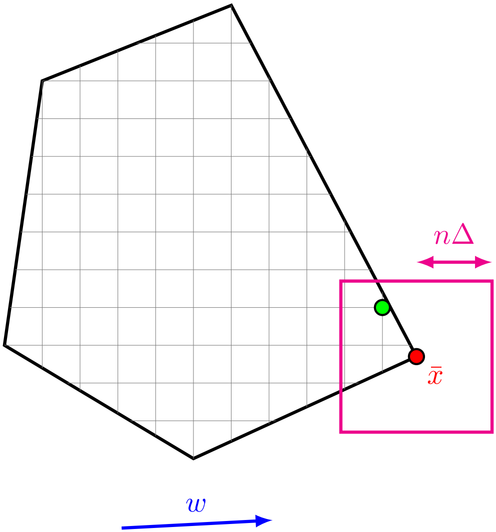
Theorem (Cook, Gerards, Schrijver, Tardos '86)

Let A be a totally Δ-modular $m \times n$ matrix and let b and w be integer vectors such that

- $Ax \leqslant b$ *has an integral solution, and*
- $\max\{w^\intercal x : Ax \leqslant b\}$ *exists.*

Then for each optimal solution $\bar{x}$ to $\max\{w^\intercal x : Ax \leqslant b\}$, there exists an optimal solution z^* to $\max\{w^\intercal x : Ax \leqslant b,\ x \in \mathbb{Z}^n\}$ with

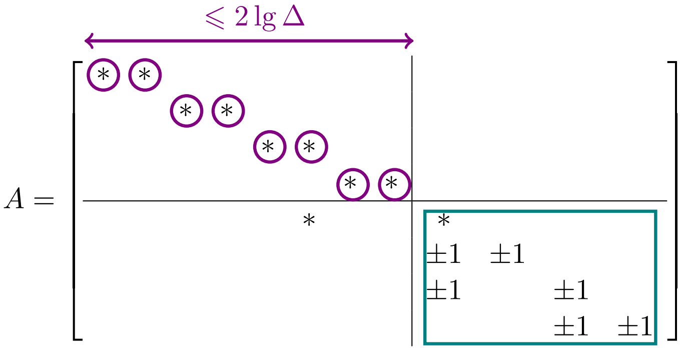
$$||\bar{x} - z^*||_\infty \leqslant n\Delta$$



$n\Delta$
$\bar{x}$
w

1st reduction: reducing to $A \in \{-1,0,1\}^{m \times n}$

After permuting rows and columns:



1st reduction:

- Solve LP relaxation $\max\{w^\intercal x : Ax \leqslant b\} \to \bar{x}$
- Guess the first $O(\lg \Delta)$ variables

2nd reduction: reducing to $A \in \{0,1\}^{m\times n}$, $b = 1$

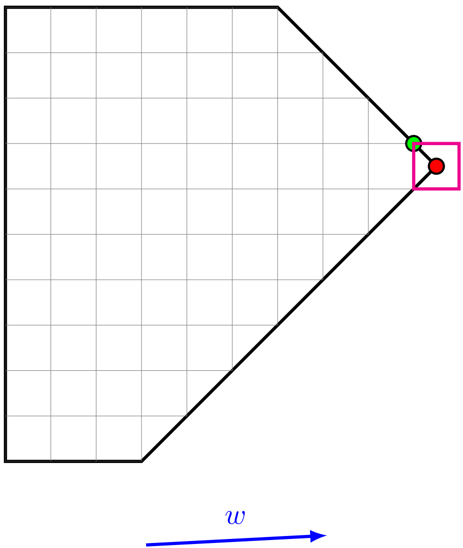
Theorem (FJWY '21)

Let $A \in \{-1,0,1\}^{m\times n}$, $b \in \mathbb{Z}^m$, $w \in \mathbb{Z}^n$. Assume that

- *every row of A has $\leqslant 2$ nonzeros,*
- *$P := \{x : Ax \leqslant b\}$ is bounded and $P \cap \mathbb{Z}^n \neq \varnothing$.*

For every extremal optimal solution $\bar{x}$ to $\max\{w^\intercal x : Ax \leqslant b\}$, there exists an opt. solution z^* to $\max\{w^\intercal x : Ax \leqslant b,\ x \in \mathbb{Z}^n\}$ with

$$||\bar{x} - z^*||_\infty \leqslant \frac{1}{2}$$


w

Final problem

After translating and reformulating, we get

$$\begin{aligned} \max \quad & w^\intercal x \\ \text{s.t.} \quad & Ax \leqslant \mathbf{1} \\ & x \in \mathbb{Z}^n \end{aligned}$$

where:

- A is the edge-vertex incidence matrix of some graph G
- $\mathrm{ocp}(G) \leqslant \lg \Delta$
- $w \in \mathrm{cone}(A^\intercal)$

Outline

Our structure theorem relies on the graph minor project of Robertson and Seymour ($\geqslant$ 23 papers, $\geqslant$ 500 pages, **'83** $\rightarrow$ **'10**)

Definition (k-DRP)

Given a graph G and k vertex pairs $(s_1, t_1), \ldots, (s_k, t_k)$, does G contain k vertex-disjoint paths $P_1, \ldots, P_k$ such that each P_i is a s_i–t_i path?

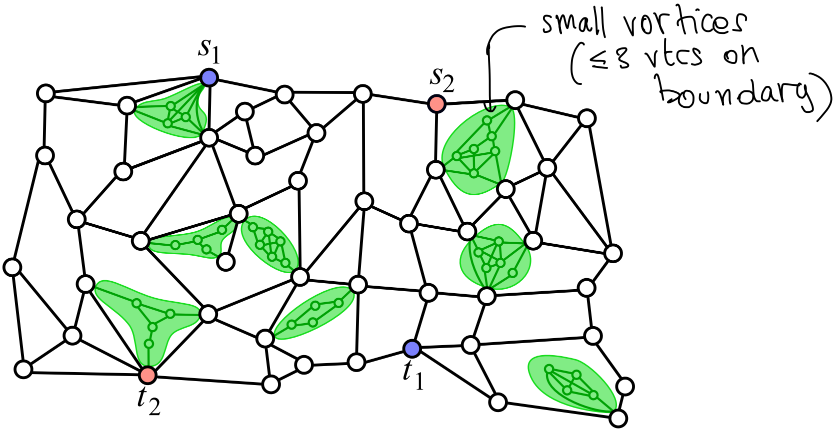
Definition (k-OCP)

Given a graph G, does G contain k vertex-disjoint odd cycles?

Remarks:

- k-DRP is a central problem the graph minor project
- both k-DRP and k-OCP have FPT algorithms
- k-DRP reduces to k-OCP

NO instances of 2-DRP / 2-OCP



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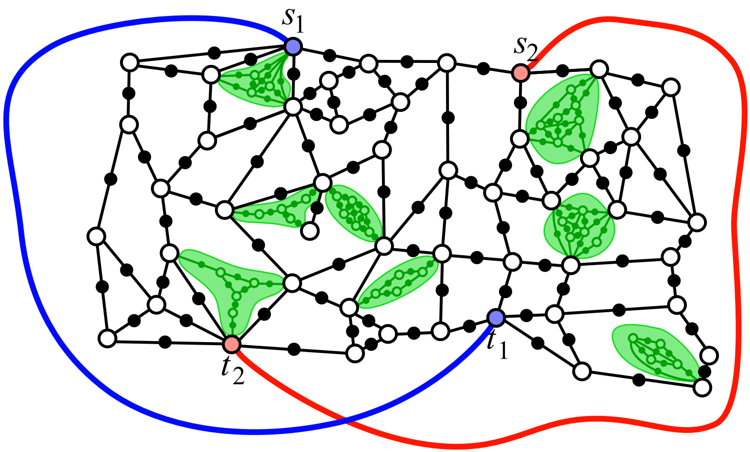
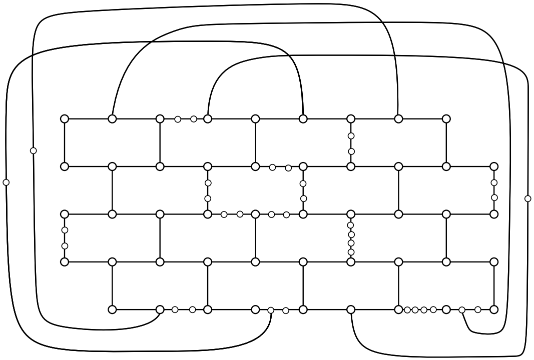


Escher walls



$\mathrm{ocp}(G) = 1, \quad \mathrm{oct}(G)$ as large as $\Omega(\sqrt{n})$

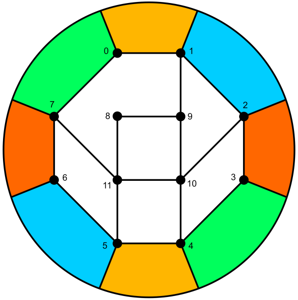
Definition

The *odd cycle transversal number* $\mathrm{oct}(G)$ is the minimum size of $X \subseteq V(G)$ such that $G - X$ is bipartite

Theorem (Lovász)

Let G be a 4-connected graph with $\mathrm{ocp}(G) \leq 1$. Then

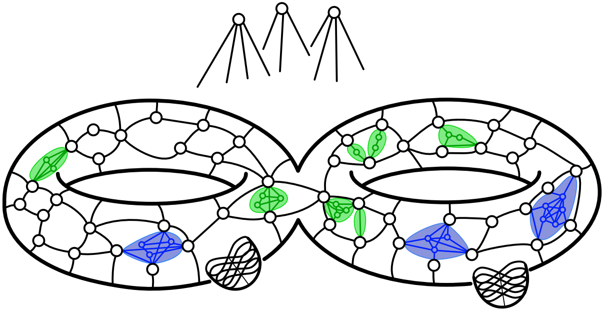
1. $\mathrm{oct}(G) \leq 3$, *or*
2. G *has an even-face embedding in the projective plane*



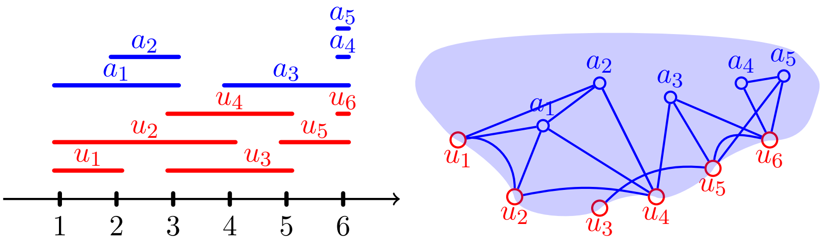
1st structure theorem

Theorem (informal)

Let $k \geq 1$ be a fixed integer. For every graph G with $\mathrm{ocp}(G) \leq k$ and $\mathrm{oct}(G)$ sufficiently large, there is a near embedding of G in a surface $\mathbb{S}$ with all parameters bounded: size of the apex set, number and adhesion of large vortices, Euler genus of $\mathbb{S}$. Moreover, the part of G embedded in $\mathbb{S}$ "essentially contains" a large Escher wall



Linear decompositions



Definition

The *adhesion* of the linear decomposition $(X_1, \ldots, X_n)$ is $\max\{|X_i \cap X_{i+1}| : i < n\}$

Toward the 2nd structure theorem

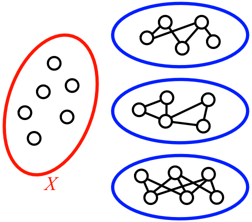
Resilience

Definition

Graph G is *ρ-resilient* if $\forall X \subseteq V(G)$ with $|X| \leqslant \rho$, $\exists$ component H of $G - X$ such that $\mathrm{ocp}(H) = \mathrm{ocp}(G)$

Remarks:

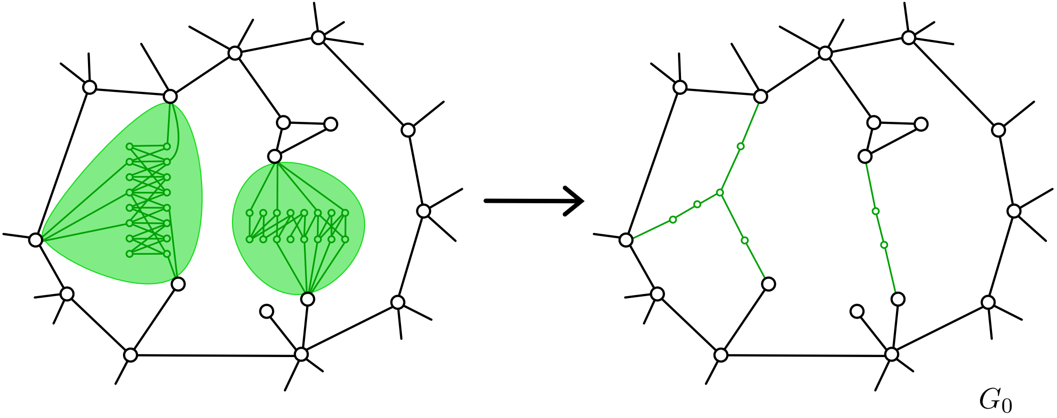
- G is *not* ρ-resilient iff $\exists X \subseteq V(G)$ with $|X| \leqslant \rho$ such that $\forall$ components H of $G - X$ have $\mathrm{ocp}(H) < \mathrm{ocp}(G)$



- For solving the stable set problem in graphs with bounded OCP, may assume that G is $\rho(k)$-resilient

Toward the 2nd structure theorem

Replacing small vortices by gadgets



2nd structure theorem

Theorem (informal)

For every integer $k \geq 1$, and for every graph G with $\mathrm{ocp}(G) \leq k$ that is sufficiently resilient, there is an near embedding in a non-orientable surface $\mathbb{S}$ with all parameters bounded, and the extra properties:

- *each small vortex is bipartite*
- *each large vortex is bipartite (even when augmented with the boundary of the face of G_0 that contains it)*
- *large vortices are vertex-disjoint*
- *every face of G_0 is bounded by a cycle*
- *every odd cycle in G_0 defines a Möbius band in $\mathbb{S}$*

Proofs of the structure theorems

Using several *graph minor papers* + own previous work:

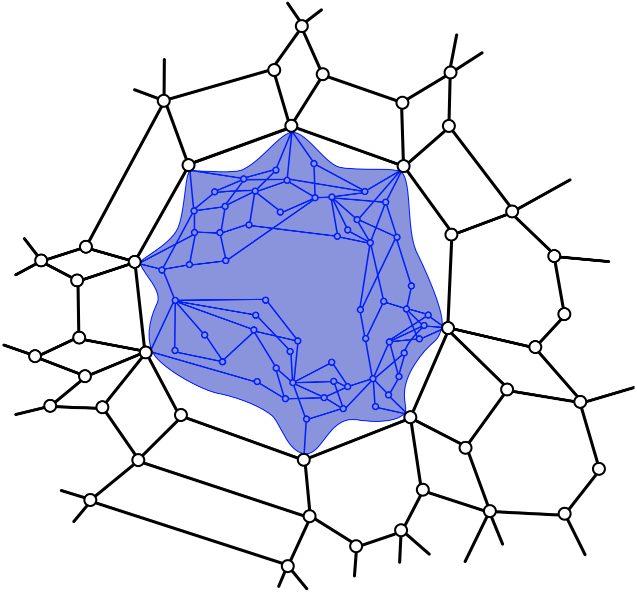
For the 1st theorem:

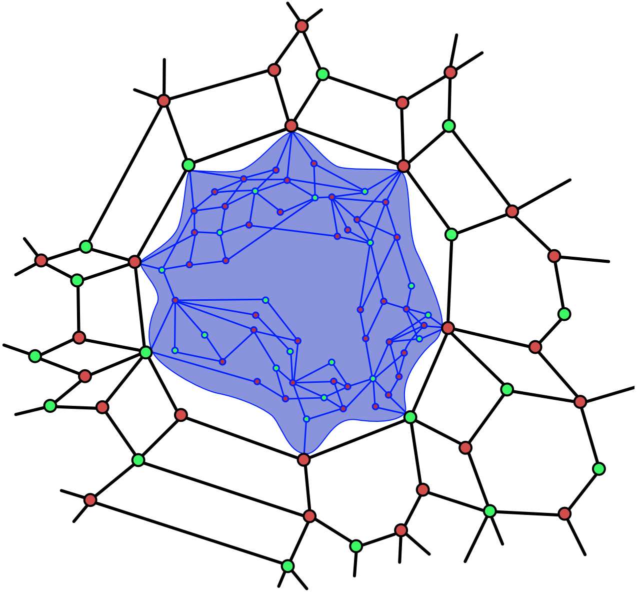
1. Reed **'99** and Kawarabayashi and Reed **'10**
2. Geelen, Gerards, Reed, Seymour, Vetta **'09**
3. Kawarabayashi, Thomas, Wollan **'20**

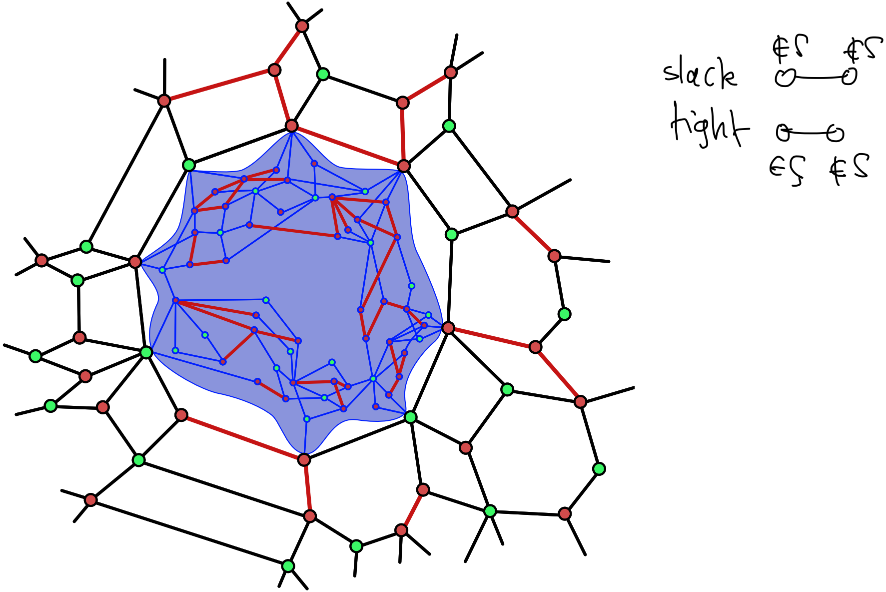
For the 2nd theorem:

1. Diestel, Kawarabayashi, Müller, Wollan **'12**
2. Conforti, F, Huynh, Weltge **'20**
3. Conforti, F, Huynh, Joret, Weltge **'21**

Outline







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slack
tight
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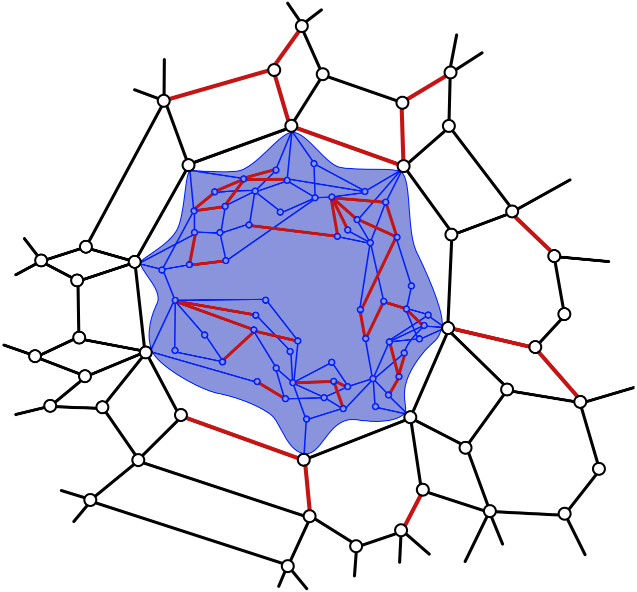


Slack vectors

Switch *vertex space* $\mathbb{R}^{V(G)} \rightarrow$ *edge space* $\mathbb{R}^{E(G)}$

Definition

Vtx weights $w \in \mathbb{R}^{V(G)}$ are *induced* by edge costs $c \in \mathbb{R}^{E(G)}_{\geqslant 0}$ if

$$w(v) = c(\delta(v)) \text{ for all } v \in V(G)$$

Definition

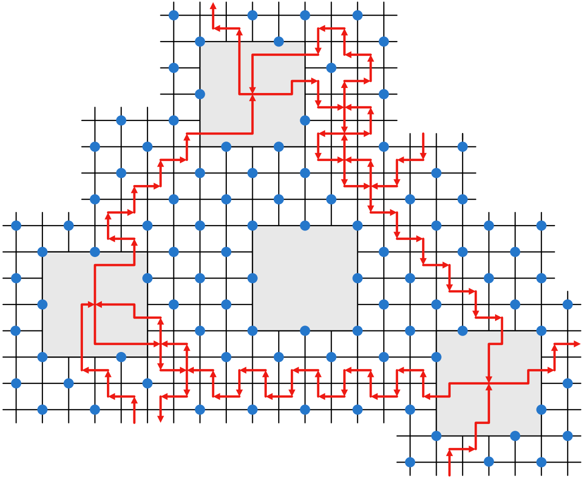
$y \in \mathbb{Z}^{E(G)}_{\geq 0}$ is a *slack vector* if

$$\exists x \in \mathbb{Z}^{V(G)} : y_{vw} = 1 - x_v - x_w \text{ for all } vw \in E(G)$$

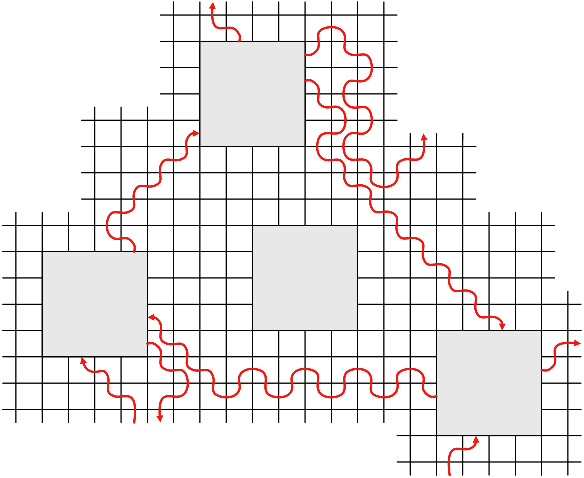
Remark:

$$w^\intercal x = c(E(G)) - c^\intercal y = \frac{1}{2} w(V(G)) - c^\intercal y$$

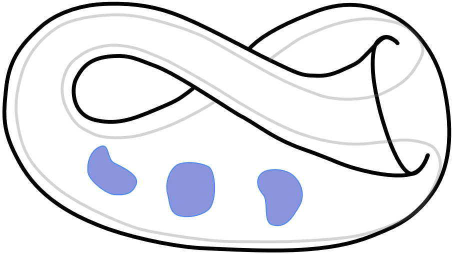
The sketch

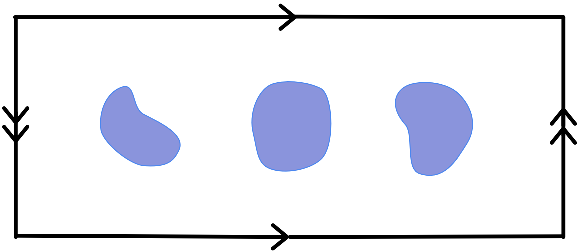


The sketch



Constructing the sketch edge by edge





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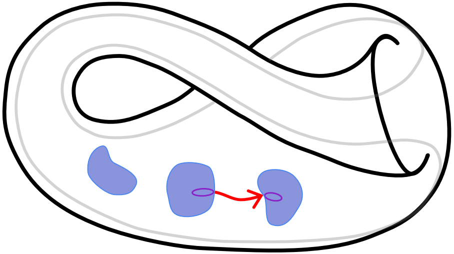

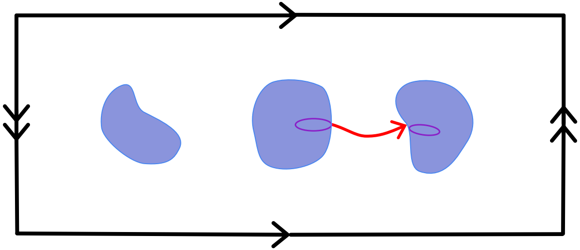

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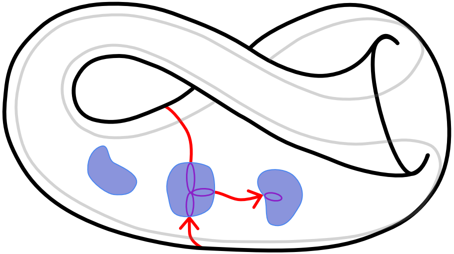

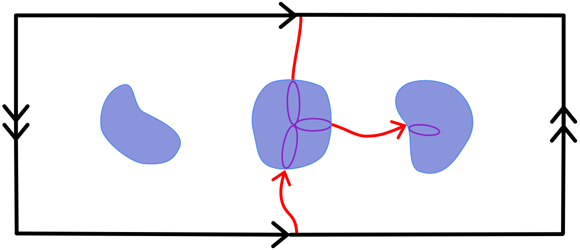

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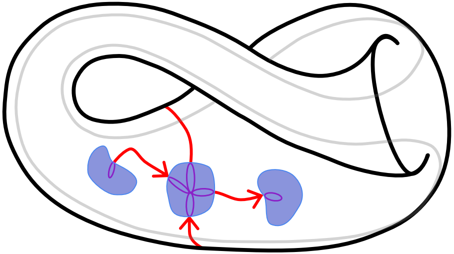

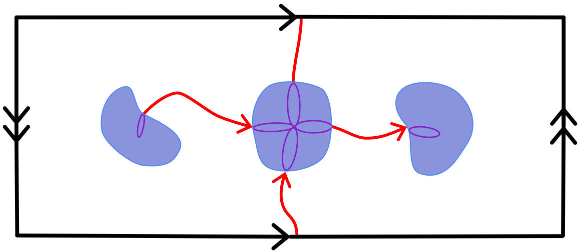

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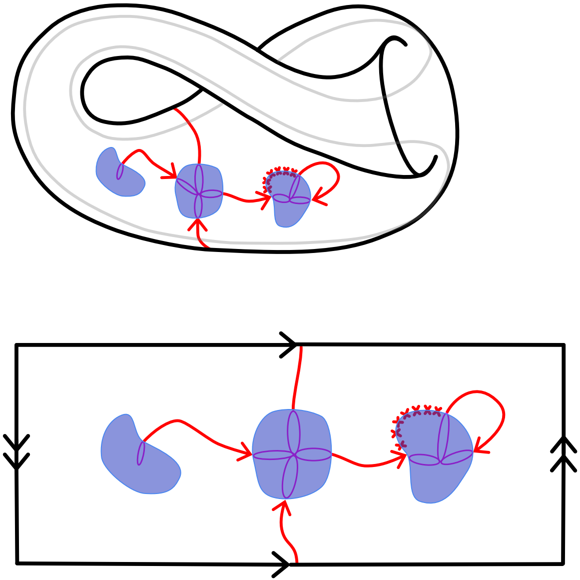

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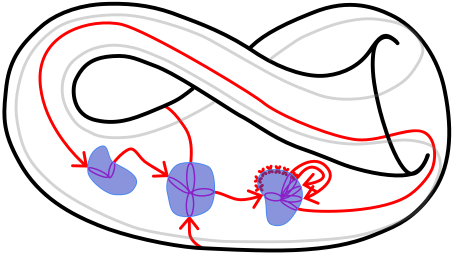

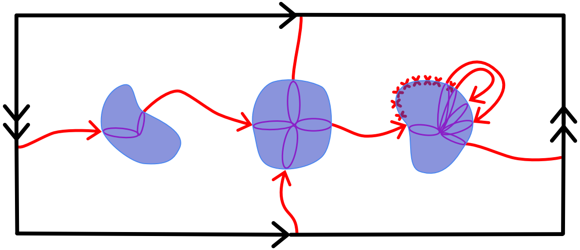

Dynamic program

Main algorithm is a dynamic program (DP):

1. Cells correspond to possible faces of the (partial) sketch
2. Use precedence rule for split operations to bound the number of cells by a polynomial
3. Every sketch edge has two corresponding cutsets, inside which the solution is guessed
4. The DP remembers “just enough” extra information to guarantee that it constructs solutions that are *feasible*

Subroutines:

- Homologous flow (Morell, Seidel and Weltge **’21**)
- Special stable set instances “between” cutsets


THANK
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