On the diameter and the circuit-diameter of polytopes

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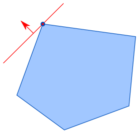

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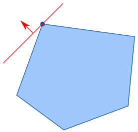

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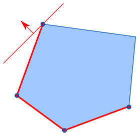

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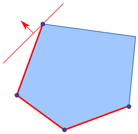

- **Simplex Algorithm's idea:** move from an extreme point to an improving adjacent one, until the optimum is found!
- The operation of moving between extreme points is called pivoting.

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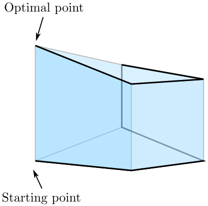

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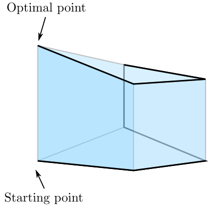

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  - Bland's rule
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...exhibiting a worst-case (sub)exponential behaviour for the Simplex algorithm [Klee&Minty'72, Jeroslow'73, Avis&Chvàtal'78, Goldfarb&Sit'79, Friedmann&Hansen&Zwick'11, Friedmann'11, Avis&Friedmann'17, Disser&Hopp'19]

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- The Simplex algorithm (with e.g. Dantzig's rule) can 'implicitly' solve hard problems [Adler,Papadimitriou&Rubinstein'14, Skutella&Disser'15, Fearnley&Savani'15]

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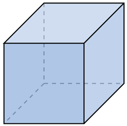
Related Question: *What is the maximum length of a 'shortest path' between two extreme points of a polytope?*

Diameter of polytopes

Diameter of polytopes

- We can naturally associate an undirected graph to a given polytope $P \subseteq \mathbb{R}^d$:
  - the vertices correspond to the extreme points of P
  - the edges are given by the 1-dimensional faces of P



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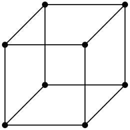

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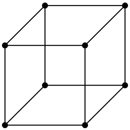

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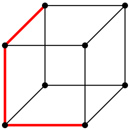

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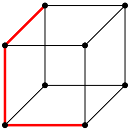

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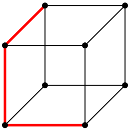

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Remark: *In order for a polynomial pivoting rule to exist, a necessary condition is a polynomial bound on the value of the diameter!*

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→ This latter result holds for half-integral polytopes with a very easy description (fractional matching polytope).

In this talk

- Characterization of the diameter of the fractional matching polytope
- Discuss hardness and algorithmic implications
- Recent generalization: circuit-diameter
- Final remarks and open questions

The matching polytope

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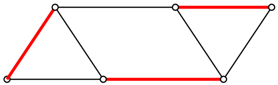

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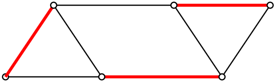
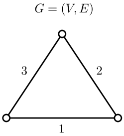
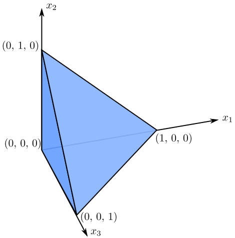


• The matching polytope ($\mathcal{P}_M$) is given by the convex hull of characteristic vectors of matchings of G.





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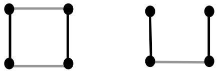

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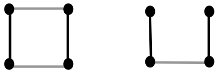

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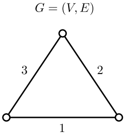

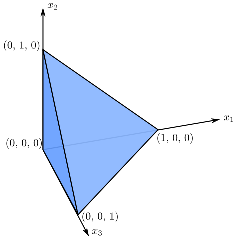

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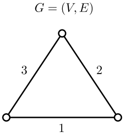

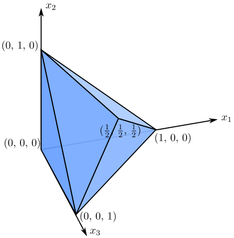

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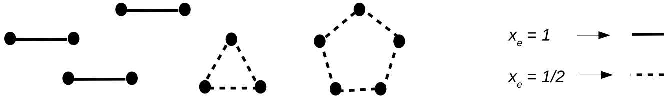

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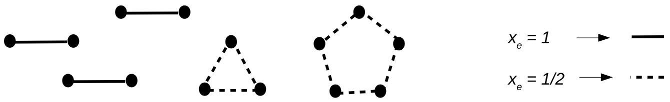

- Our characterization of the diameter is as follows:

Theorem [S.'18]

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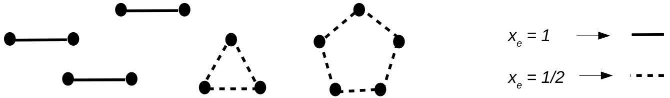

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The fractional matching polytope

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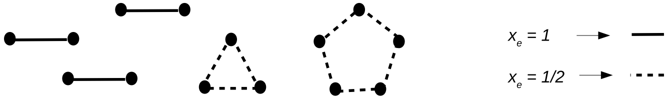

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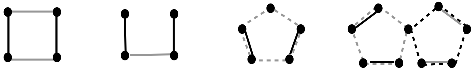

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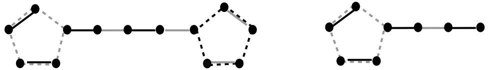

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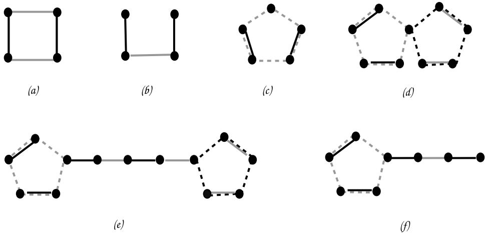

Note: In all cases, $\Big(\#\text{ of degree-1 nodes} + \#\text{ of odd cycles}\Big) \leq 2$

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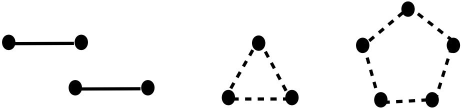

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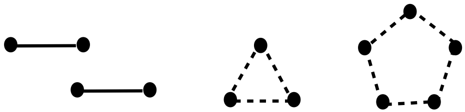

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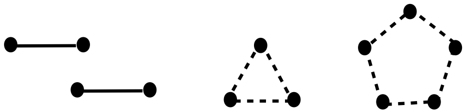

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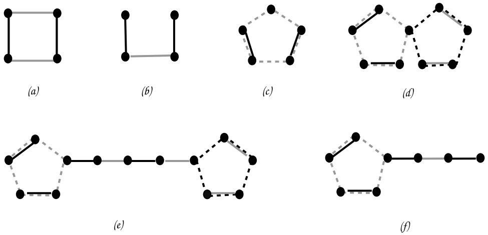

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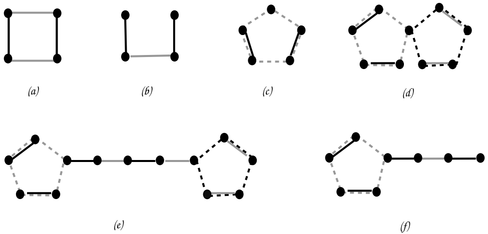

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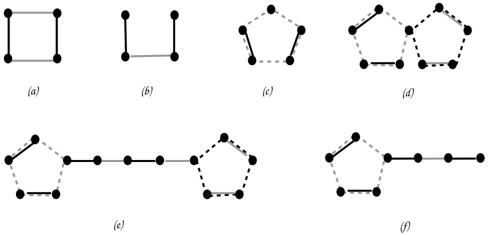

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...but unfortunately this may lead to paths longer than the claimed bound!

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- Given two distinct vertices z and y of $\mathcal{P}_{FM}$, we
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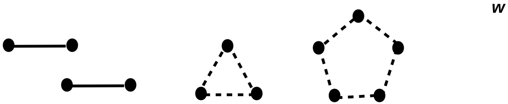

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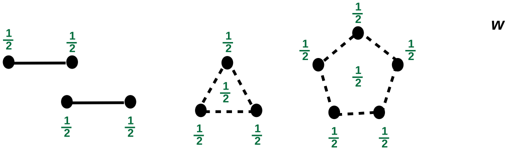

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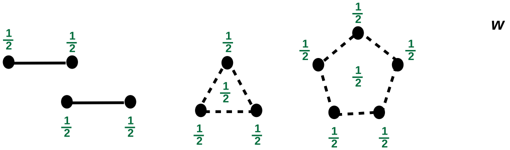

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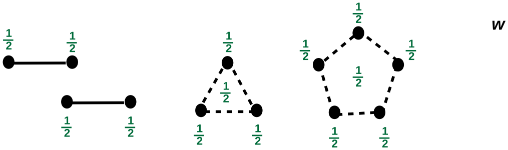

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- With some extra effort, we can strengthen the result to show APX-hardness.

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- **Consequences** (unless P=NP):
  - For any efficient pivoting rule, an edge-walking algorithm (like Simplex) can't reach the optimum with a min number of augmentations.

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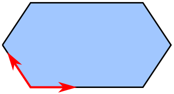

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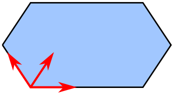

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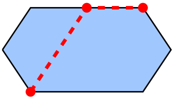

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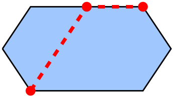

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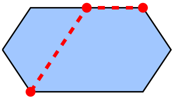

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- [Borgwardt,Finhold,Hemmecke'14] formalized the notion of circuit-diameter.

  Circuit-diameter: max-value of a shortest path between two extreme points, assuming that at any given point we can move *maximally* along *any* circuit.

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Corollary [De Loera, Kafer, S.'19]

There exists a polynomial function $f(m, \alpha)$ that bounds the circuit-diameter of any rational polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b, Bx \leq d\}$ with m constraints and maximum encoding length of a coefficient equal to α.

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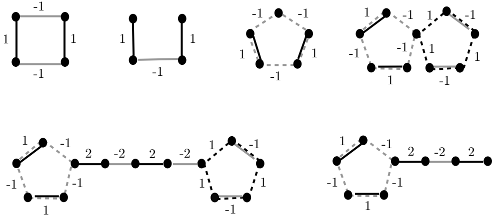

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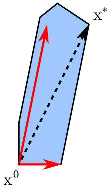

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Proof idea:

- Main ingredient: Showing that moving along a steepest-edge direction yields an n-approximation of moving along a greatest-improvement circuit.
- Improve the analysis relying on the technique of [Frank,Tardos'87], to make the above number strongly polynomial. □

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