Optimality and uniqueness of the D_4 root system

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Joint works with Henry Cohn, David de Laat and Willem de Muinck Keizer

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October 2024

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- Kissing number problem

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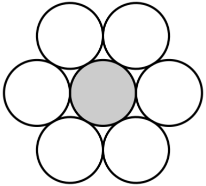
The kissing number problem



The kissing number $k(n)$ is the maximum number of unit spheres that simultaneously touch a central unit sphere.

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- Q: can something similar as in dimension 3 happen?
- A: No, we prove that the D_4 root system is the unique optimal kissing configuration in dimension 4, and is an optimal spherical code.

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The Lasserre hierarchy for packing problems

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- In dimension $n = 8$ and $n = 24$, $\mathrm{las}_1(n) = k(n)$ (Delsarte LP bound).
- We prove that $\mathrm{las}_2(4) = k(4) = 24$ by giving an exact solution.

The independent set problem

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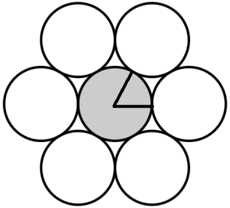

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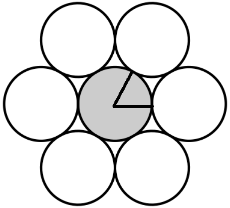

Take $V = S^{n-1}$ and distinct $x, y \in V$ adjacent if $x \cdot y > 1/2$. Then the kissing number is the maximum size of an independent set in this graph.

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Then the t-th step of the Lasserre hierarchy is given by:

$$\begin{array}{lll} \text{minimize} & K(\emptyset, \emptyset) & \\ \text{subject to} & A_t K(S) \leq -\chi_{\mathcal{I}_{=1}}(S) & \forall\, S \in \mathcal{I}_{2t} \setminus \{\emptyset\} \\ & K \in C(\mathcal{I}_t \times \mathcal{I}_t)_{\succeq 0} & \end{array}$$

D. de Laat and F. Vallentin. "A semidefinite programming hierarchy for packing problems in discrete geometry". en. In: *Mathematical Programming* 151.2 (July 2015). arXiv: 1311.3789, pp. 529–553. ISSN: 0025-5610, 1436-4646. DOI: 10.1007/s10107-014-0843-4. URL: http://arxiv.org/abs/1311.3789 (visited on 10/26/2021).

Proof that las_t gives an upper bound

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so $K(\emptyset,\emptyset) \geq |C|$.

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So sharp bounds give information about possible configurations!

Optimality and uniqueness of the D_4 root system

By complementary slackness, our exact optimal solution to $\mathrm{las}_2(4)$ shows that for all sets C of size 24,

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- the D_4 root system is the unique optimal spherical code with 24 points in dimension 4.

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Rounding SDP solutions

What is rounding?

$$\begin{aligned} \text{maximize} \quad & \langle C, X \rangle \\ \text{subject to} \quad & \langle A_i, X \rangle = b_i, \quad i = 1, \ldots, m \\ & X \succeq 0 \end{aligned}$$

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A solver gives $X^* \succeq -\varepsilon I$ such that

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Assumptions:

- Everything can be done over algebraic fields of low degree (in this talk we use $\mathbb{Q}$ for convenience)
- There is a basis of the kernel of an optimal solution X of small bitsize

The rounding procedure

Procedure:

1. Find the kernel vectors

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2. Then we modify the remaining eigenvectors and eigenvalues a little bit to satisfy the constraints, so that strictly positive eigenvalues stay strictly positive.

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Transformations

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So the optimal face is described by

$$\hat{F} = \{\widehat{X} \succeq 0 : \langle \widehat{A}_i, \widehat{X} \rangle = b_i, i = 1, \dots, m\}.$$

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This is both fast and gives solutions of small size.

Questions?

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The rounding procedure is available as part of the Julia SDP solver `ClusteredLowRankSolver.jl`.