# Two Complexity Results about Polynomial Optimization and Lasserre hierarchies

**Luis Felipe Vargas,**

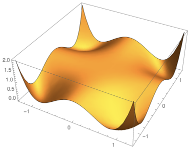
*Dutch Optimization Seminar*

*Joint work with Monique Laurent*

September, 2023

## Polynomial Optimization

Positive polynomials
vs.
Sums of squares of polynomials

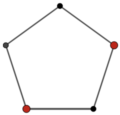



## Graph theory

Graph parameters

Stability number of a graph

5-cycle $C_5$



# Polynomial optimization

Given a polynomials $f$ and $g_1, \ldots, g_m$, a polynomial optimization problem (PoP) is:

$$f^* = \inf_{x \in K} f(x), \tag{PoP}$$

where

$$K = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \text{ for } i = 1, \ldots, m\}$$

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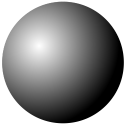
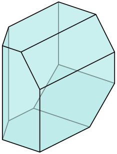
- Polynomial equations $p(x) = 0$ can be added $(p(x) \geq 0, p(x) \leq 0)$.
- Solving (PoP) is very hard in general.

# Examples of semialgebraic sets $K$



Sphere: $\{x \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 \leq 1\}$



Polytopes: Linear inequalities

Nonconvex in general: $\{1 \leq x^2 + y^2 \leq 4\}$

$\{0,1\}^n$ Discrete sets $x_i^2 = x_i$, for $i \in [n]$

# Sum-of-squares approximations

$$K = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \text{ for } i = 1, \ldots, m\}$$

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"is a sum of squares"

**Definition.** A polynomial $p$ is a **sum of squares (SOS)** if

$$p = q_1^2 + q_2^2 + \cdots + q_m^2$$

for some polynomials $q_i$.

If $f$ is SOS, then $f(x) \geq 0$ for all $x \in \mathbb{R}^n$.

# Why sums of squares?

Sum-of-squares $\longleftrightarrow$ PSD matrix

is $p = x^4 - 2x^3y + 2x^2y^2 - 2x^2y - 2x + 1$ a sum of squares?

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Yes

$p = (x^2 - xy)^2 + (x - 1)^2 + (x - xy)^2$

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$p = (x^2 - xy)^2 + (x-1)^2 + (x-xy)^2$ Let $m^t = (1, x, y, x^2, xy, y^2)$

$$P = \overbrace{m^t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}}^{p_1} \overbrace{(0\,0\,0\,1\,-1\,0)}^{p_1} m + \overbrace{m^t \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}^{p_2} \overbrace{(-1\,1\,0\,0\,0\,0)}^{p_2} m + \overbrace{m^t \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}}^{p_3} \overbrace{(0\,1\,0\,0\,-1\,0)}^{p_3} m$$

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$$\boxed{p \text{ is SOS} \iff p = m^t Q m, \text{ for some } Q \succeq 0}$$

- The constrain "$p$ is a sum of squares" can be modeled via a semidefinite program.

# Certificates using sums of squares: On semialgebraic sets

Let $K = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, \dots, g_m(x) \geq 0\}$

The quadratic module defined by $\boldsymbol{g}$ is

$$M(\boldsymbol{g}) = \left\{\sigma_0 + \sigma_1 g_1 + \cdots + \sigma_m g_m : \sigma_i \text{is SOS}\right\}$$

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**Theorem (Putinar)**

*Assume the archimedean conditions holds: $N - \sum_{i=1}^n x_i^2 \in M(\boldsymbol{g})$, for some $N \in \mathbb{N}$.*

$$\textit{If } f > 0 \textit{ on } K, \textit{ then } f \in M(\boldsymbol{g})$$

- The archimedean condition implies that $K$ is compact.
- The condition $f > 0$ is necessary in general.

# Lasserre hierarchy for polynomial optimization

$K = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$

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Define the truncated quadratic module

$$M(\boldsymbol{g})_r = \left\{ \underbrace{\sigma_0}_{\deg \leq 2r} + \underbrace{\sigma_1 g_1}_{\deg \leq 2r} + \cdots + \underbrace{\sigma_m g_m}_{\deg \leq 2r} : \sigma_i \text{is SOS} \right\}$$

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We say that we have finite convergence if $f^{(r)} = f^*$ for some $r$.

- It is not always achieved.

# When do we have finite convergence?

$$f^* = \inf_{x \in K} f(x), \qquad \text{(PoP)}$$

$$K = \{x \in \mathbb{R}^n : g_i(x) \geq 0, h_j(x) = 0 \text{ for } i \in [m], j \in [l]\}$$

Various results by Lasserre, Marshall, Nie, Scheiderer, Schweighofer.

1. When $V_{\mathbb{R}}(h)$ is finite. [Nie, 2012]
2. When (PoP) has finitely many minimizers and they satisfy the classical optimality conditions. [Nie, 2014]
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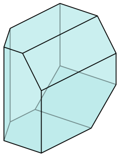
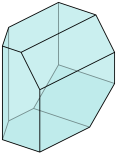
**In this talk**: What is the complexity of

1. Deciding whether (PoP) has finitely many minimizers?
2. Deciding whether the Lasserre hierarchy of (PoP) has finite convergence?

# Complexity questions: Linear case



Let (L-P) be a linear program



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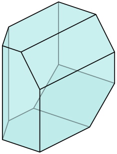

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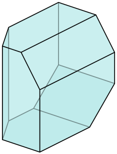

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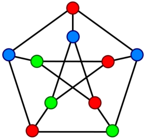
**Finite convergence?**

- Yes, always. The Lasserre hierarchy (at $r = 1$) finds the optimal solution.

# Stability number of a graph

Let $G = (V, E)$ be a graph.



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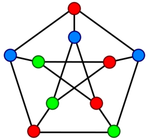

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**Discrete formulation**

$$\alpha(G) = \max\left\{ \sum_{i \in V} x_i : x_i^2 = x_i \text{ for } i \in V, x_i x_j = 0 \text{ for } \{i,j\} \in E \right\}$$

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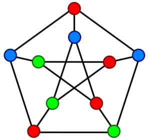

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**Motzkin-Straus 1965**

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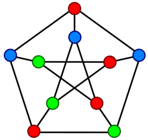

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When do we have finite convergence?

$$\boxed{\frac{1}{\alpha(G)} = \min\left\{x^T(A_G + I)x : \sum_{i=1}^n x_i = 1, x \geq 0\right\}}$$

For any $S$ stable of size $\alpha(G)$, $x = \frac{1}{\alpha(G)}\chi^S$ is a minimizer:

$$\frac{1}{\alpha(G)} \cdot \underbrace{(1\,1 \cdots 1}_{S}\; 0 \cdots 0) \begin{pmatrix} 1 & & 0 & \\ & 1 & & \\ 0 & & \ddots & \\ & & & 1 \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdot \frac{1}{\alpha(G)} = \frac{1}{\alpha(G)^2} \cdot \alpha(G) = \frac{1}{\alpha(G)}.$$

# Motzkin-Straus Formulation

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Consider $G = C_5$ the 5-cycle. Then, for any $t \in [0,1]$ we have:

$$\frac{1}{2} \cdot (t, 1-t, 0, 1, 0) \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} t \\ 1-t \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot \frac{1}{2} = \frac{(t + (1-t))^2 + 1}{4} = \frac{1}{2}.$$

# Role of Critical Edges

**Definition.** An edge $e$ of $G$ is **critical** if $\alpha(G \setminus e) = \alpha(G) + 1$.

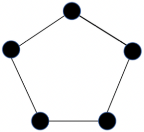
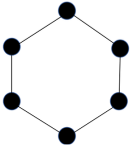


**Figure 1:** $C_5$, all edges are critical



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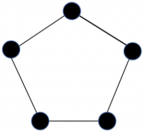

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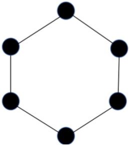

**Figure 2:** $C_6$, no edge is critical

**Theorem**

*Given a graph $G$ and an edge $e$. The problem of deciding whether $e$ is critical in $G$ is NP-hard.*

**Theorem (Minimizers of (M-S))**

*Let $x$ be feasible for (M-S) with support $S := \{i : x_i > 0\}$, and $C_1, C_2, \dots, C_k$ the connected components of the graph $G[S]$. Then $x$ is an optimal solution of (M-S) if and only if the following holds:*

- $k = \alpha(G)$,
- $C_i$ *is a clique for all* $i \in [k]$,
- $\sum_{j \in C_i} x_j = \frac{1}{\alpha(G)}$ *for all* $i \in [k]$.

*In that case, all edges of* $G[S]$ *are* **critical.**

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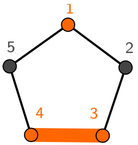


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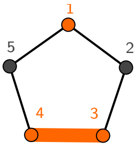


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The **only** edges in the support of an optimal solution are **critical**.

# Perturbed Motzkin-Straus formulation

For an edge $e \in E$, consider the following problem

$$\frac{1}{\alpha(G)} = \min x^T(A_G + I + A_{G\setminus e})x \text{ subject to } \sum_{i=0}^{n} x_i = 1, x \geq 0 \qquad (1)$$

The optimal value is $\frac{1}{\alpha(G)}$ as $x = \frac{1}{\alpha(G)}\chi^S$ is a solution.

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*The problem of deciding whether a polynomial optimization problem (even quadratic over the simplex) has finitely many minimizers is NP-hard*

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**Corollary**

*The problem of deciding whether the Lasserre hierarchy of a polynomial optimization problem has finite convergence is NP-hard.*

# Final remarks

## Summary

We show NP-hardness of:

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Main tools:

- Motzkin-Straus formulation (and perturbations of it)
- Critical edges.

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## Related work

- A. Ahmadi and Zhang have used the Motzkin-Straus formulation for obtaining complexity results in optimization (local minmizers, .. )