Extremal lattice problems (not in the bible)

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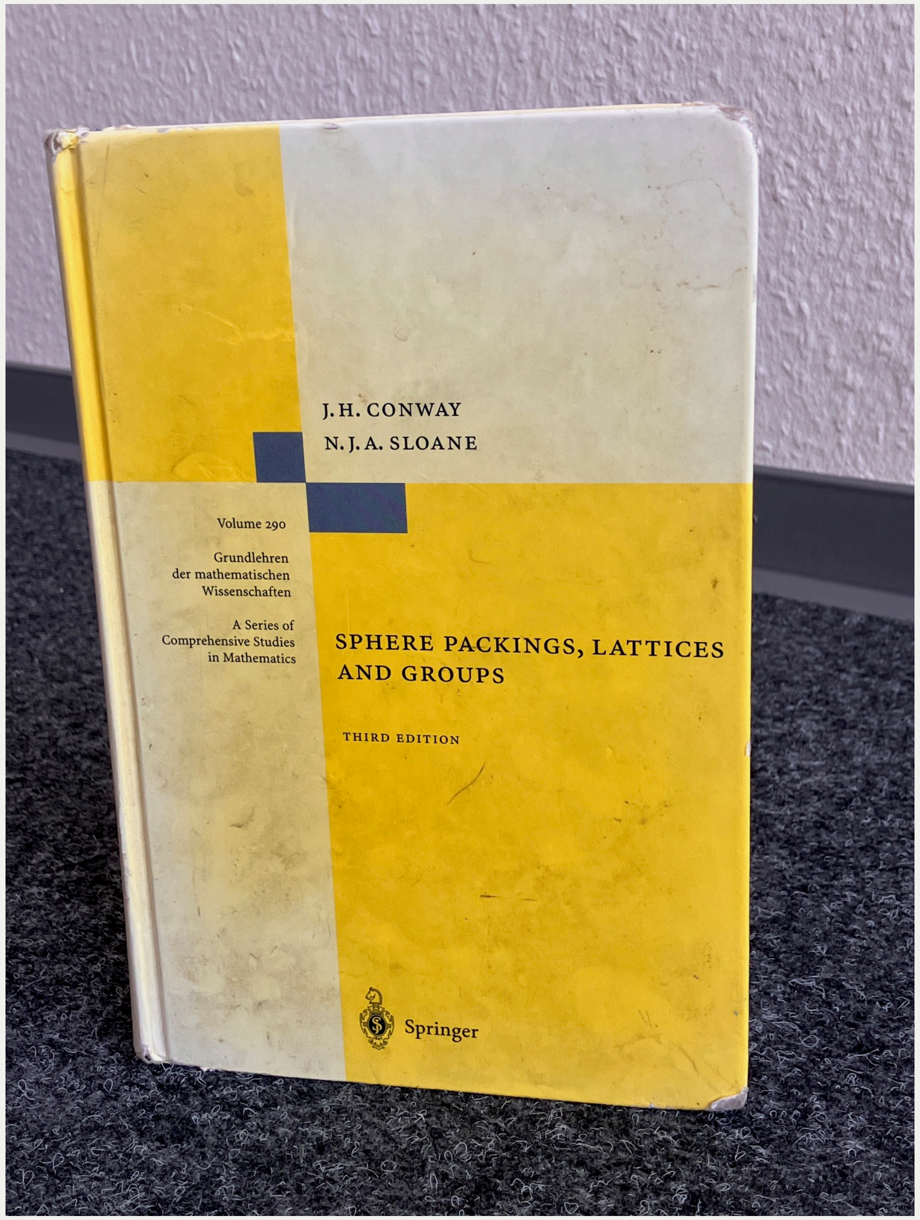
SPLAG - The „bible“



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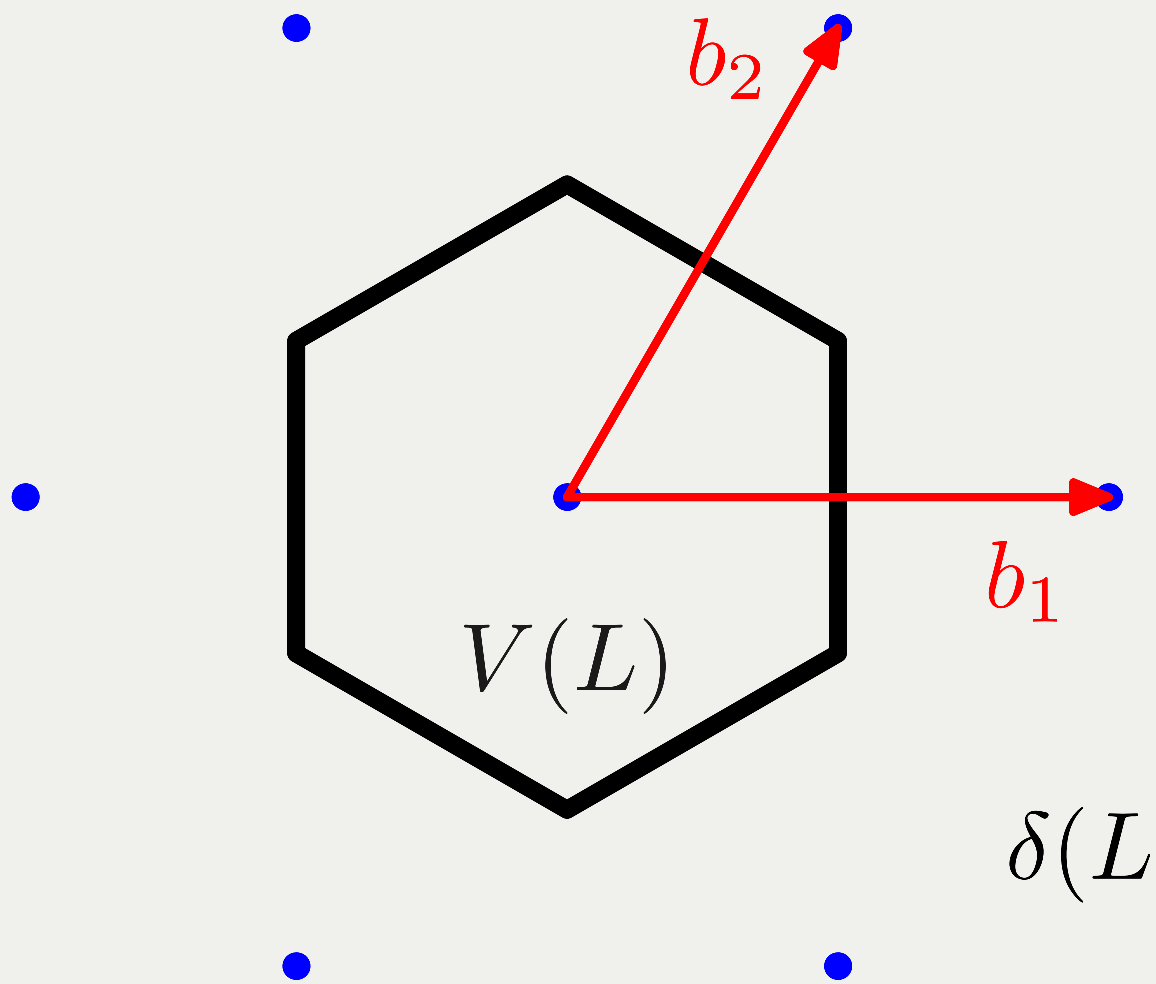
[1]We also thank the correspondent who reported hearing the first edition described during a talk as “the bible of the subject, and, like the bible, [it] contains no proofs”. This is of course only half true.

Lattices

$b_1, \ldots, b_n$ basis of Euclidean space E

$L = \mathbb{Z}b_1 + \cdots + \mathbb{Z}b_n$ lattice



$V(L)$ = Voronoi cell

vol L = volume of $V(L)$

$$= \sqrt{|\det(b_1, \ldots, b_n)|}$$

$$\delta(L) = \mathrm{vol}(L)^{-1} \text{ point density of } L$$

Some optimization problems with lattices

- lattice sphere packing
- lattice sphere covering

- *coloring the Voronoi cells*
- *potential energy minimization*
- max-min polarization
- minimizing Euclidean distortion

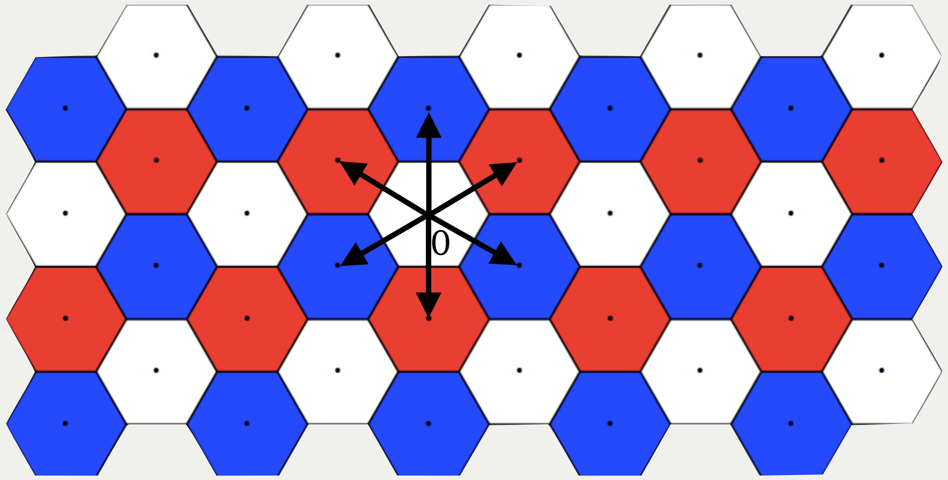
Coloring the Voronoi cells

$\Lambda \subseteq \mathbb{R}^n$ lattice

$V(\Lambda, v) = \{x \in \mathbb{R}^n : \|x - v\| \leq \|x - w\| \; \forall w \in \Lambda\}, v \in \Lambda$ Voronoi tessellation

Voronoi vectors

$v \in \text{Vor}(\Lambda) \iff V(\Lambda, 0) \cap V(\Lambda, v)$ is $(n-1)$-dim. facet



Cayley(Λ, Vor(Λ)) graph with vertices V and edges E

vertices: $V = \Lambda$, edges: $\{v, w\} \in E \iff v - w \in \text{Vor}(\Lambda)$

Determine chromatic number of this infinite graph:

$\chi(\Lambda) = \chi(\text{Cayley}(\Lambda, \text{Vor}(\Lambda)))$?

Coloring the Voronoi cells: Root lattices

$\Lambda \subseteq \mathbb{R}^n$ root lattice: $\forall v, w \in \Lambda : v \cdot w \in \mathbb{Z}$
$\forall v \in \Lambda : v \cdot v \in 2\mathbb{Z}$
$R(\Lambda) = \{v \in \Lambda : v \cdot v = 2\}$ generates Λ

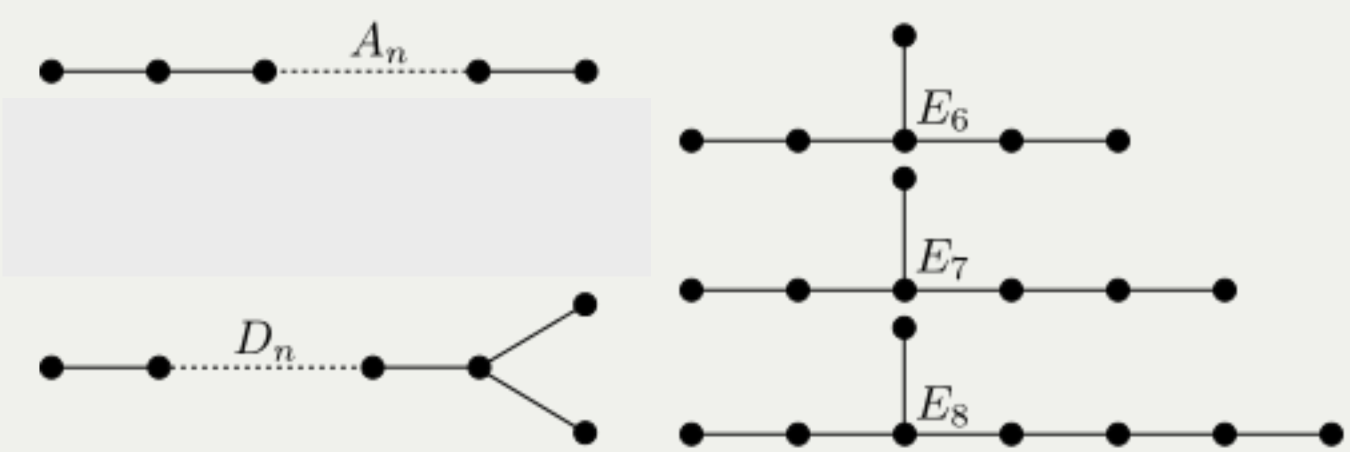
If Λ is a root lattice, then $R(\Lambda) = \mathrm{Vor}(\Lambda)$.

Classification of Witt (1941):

Every root lattice is orthogonal direct sum of irreducible root lattices.

The irreducible root lattices are A_n, D_n, E_6, E_7, E_8.

Coxeter-Dynkin diagrams of irreducible root lattices:



b_i, b_j connected iff $b_i \cdot b_j = -1$

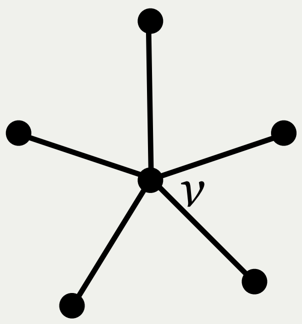
b_i, b_j not connected iff $b_i \cdot b_j = 0$

Coloring the Voronoi cells: Spectral bound I

Hoffman bound: Let $G = (V, E)$ r-regular graph
then, $\chi(G) \geq 1 - \frac{1}{m(A)}$.

$A \in \mathbb{R}^{V \times V}$ normalized adjacency operator

$$Af(v) = \frac{1}{r} \sum_{\substack{w \in V \\ \{v,w\} \in E}} f(w)$$



$m(A) = \min_{\|f\|=1} (Af, f)$ smallest eigenvalue of A

Bachoc, DeCorte, Oliveira Vallentin (2014): also works for infinite graphs

$$A : \ell^2(\Lambda) \to \ell^2(\Lambda)$$

$$Af(v) = \frac{1}{|\mathrm{Vor}(\Lambda)|} \sum_{u \in \mathrm{Vor}(\Lambda)} f(v - u)$$

$$\ell^2(\Lambda) = \left\{ f : \Lambda \to \mathbb{C} : \sum_{v \in \Lambda} |f(v)|^2 < \infty \right\}$$

Coloring the Voronoi cells: Spectral bound 2

Dutour Sikirić, Madore, Moustrou, Vallentin (2021):

Theorem. $\chi(\Lambda) \geq 1 - \left(\inf_{x \in \mathbb{R}^n/\Lambda^*} \frac{1}{|\mathrm{Vor}(\Lambda)|} \sum_{u \in \mathrm{Vor}(\Lambda)} e^{2\pi i u \cdot x} \right)^{-1}.$

irred. root lattice	*spectral lower bound*	*exact value*
A_n	$n+1$	$n+1$
D_n	n, when n even	$\chi(\frac{1}{2}H_n)$
	$n+1$, when n odd	$\chi(\frac{1}{2}H_n)$
E_6	9	9
E_7	10	14
E_8	16	16

$\frac{1}{2}H_n = \mathrm{conv}\{x \in \{0,1\}^n : \sum_i x_i = 0 \mod 2\}$ parity polytope

Serre's Oberwolfach report (12/2004)

Coloring the Voronoi cells: Open questions

- Is there a *finite* algorithm to determine the chromatic number?

- Can one define a *chromatic "polynomial"*?

Potential energy minimization

consider **potential function** $p : (0, \infty) \to \mathbb{R}$

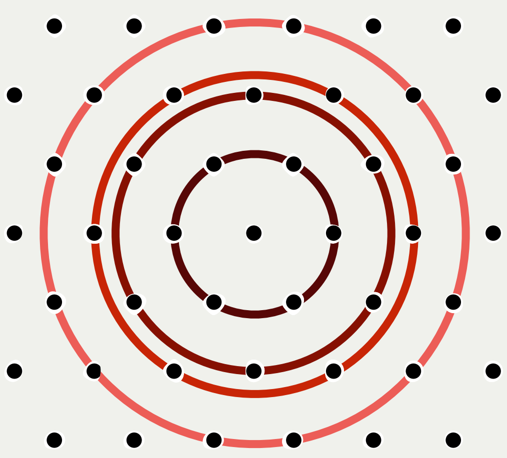
like **inverse power laws** $p(r) = \frac{1}{r^s}$, $s > 0$

or **Gaussians** $p(r) = e^{-\alpha r^2}$, $\alpha > 0$

p-potential energy of lattice L

$$\mathcal{E}(p, L) = \liminf_{r\to\infty} \frac{1}{|L \cap rB_n|} \sum_{x,y\in L\cap rB_n, x\neq y} p(|x-y|).$$

$$= \sum_{x\in L\setminus\{0\}} p(|x|)$$



Groundstate lattices $\min_{L\ n\text{-dim. lattice}, \delta(L)=1} \mathcal{E}(p, L)$

Gaussian core model

restrict p to be Gaussian $p(r) = e^{-\alpha r^2}$

Theorem (Bernstein, 1928). Exponentials $r \mapsto e^{-\alpha r}$, $\alpha \geq 0$, form extreme rays of **cone of completely monotonic functions**

$g : (0, \infty) \to \mathbb{R}$ **completely monotonic** if $(-1)^k g^{(k)} \geq 0$ for all $k \geq 0$.

If g is completely monotonic, then there is a measure μ so that

$$g(r) = \int_0^\infty e^{-\alpha r} \, d\mu(\alpha).$$

Universal optimality - Global results

Cohn, Kumar (2006)

L lattice is **universally optimal** if L minimizes $\mathcal{E}(p, L)$ among all n-dimensional lattices with $\delta(L) = 1$ and for all p where $p(|x|) = g(|x|^2)$ for some completely monotonic function g.

$L = \mathbb{Z}$ is universally optimal

Cohn-Kumar Conjecture: $L = A_2, E_8, \Lambda_{24}$ are universally optimal

Cohn, Kumar, Miller, Radchenko, Viazovska (2019): Resolve $L = E_8, \Lambda_{24}$

Bernstein $\Longrightarrow$ suffices to consider only Gaussian potential functions

Local methods: Gradients and Hessians

Coulangeon (2006)

Consider Gaussian $f_\alpha(r) = e^{-\alpha r^2}$, $\alpha > 0$.

gradient

$$\langle \nabla \mathcal{E}(f_\alpha, L), H \rangle = -\alpha \sum_{x \in L \setminus \{0\}} H[x] e^{-\alpha \|x\|^2}.$$

H n-dimensional symmetric matrix, $\mathrm{Tr}\ H = 0$

$H[x] = x^{\mathsf{T}} H x \qquad \langle A, B \rangle = \mathrm{Tr}\ AB$

Hessian

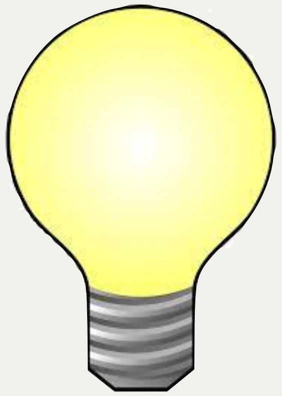
$$\nabla^2 \mathcal{E}(f_\alpha, L)[H] = \alpha \sum_{x \in L \setminus \{0\}} e^{-\alpha \|x\|^2} \left(\frac{\alpha}{2} H[x]^2 - \frac{1}{2} H^2[x] \right).$$

Question

Regev, Stephen-Davidowitz (2020)

Are there lattices which are **local maxima** for f_α-potential energy?

We suspect "Yes", and one can find them among the Niemeier lattices.



Heimendahl, Marafioti, Thiemeyer, Vallentin, Zimmermann (IMRN, 2023)

gradient

$$\langle \nabla \mathcal{E}(f_\alpha, L), H \rangle = -\alpha \sum_{x \in L \setminus \{0\}} H[x] e^{-\alpha \|x\|^2}.$$

Hessian

$$\nabla^2 \mathcal{E}(f_\alpha, L)[H] = \alpha \sum_{x \in L \setminus \{0\}} e^{-\alpha \|x\|^2} \left(\frac{\alpha}{2} H[x]^2 - \frac{1}{2} H^2[x] \right).$$

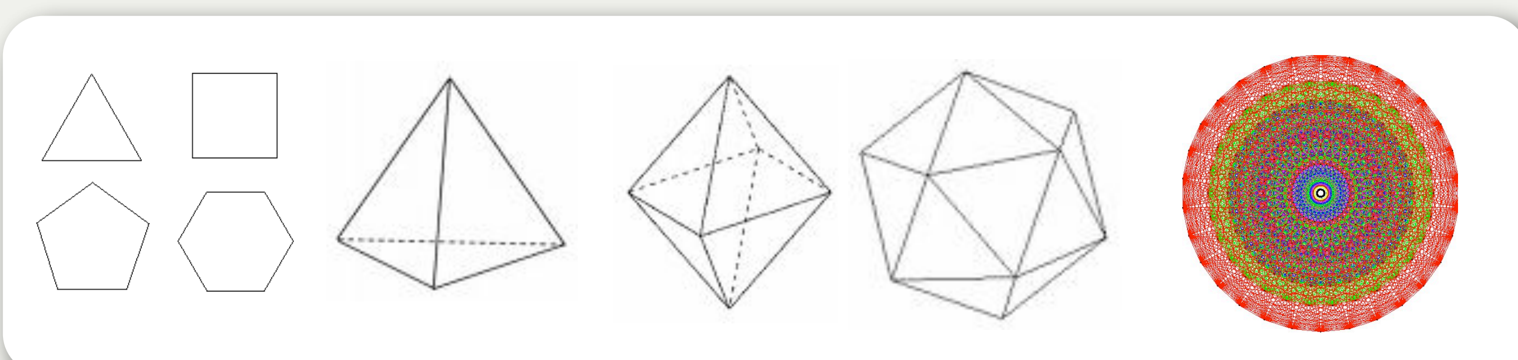
Tool from geometry: Spherical designs

$X \subseteq S^{n-1}(r) = \{x \in \mathbb{R}^n : \|x\| = r\}$ forms a spherical t-design

$$\int_{S^{n-1}(r)} p(x)\, dx = \frac{1}{|X|} \sum_{x \in X} p(x)$$

for all polynomials p of degree $\leq t$.

$$\iff \sum_{x \in X} p(x) = 0 \text{ for all } p \text{ with } \Delta p = 0,\ \deg p = 1, \ldots, t.$$



point conf.	t
n-gon	n - 1
simplex	2
cross polytope	3
icosahedron	5
240	7
196560	11

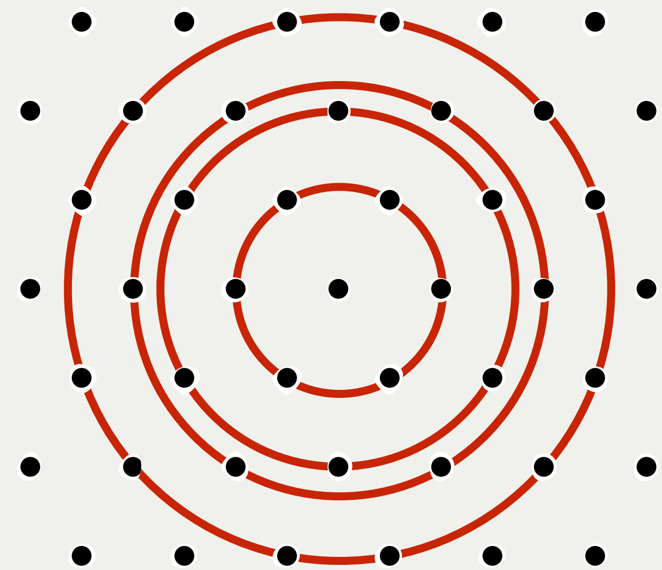
Spherical designs and criticality

If $X \subseteq S^{n-1}(r)$ forms a spherical 2-design, then

$$\sum_{x \in X} x x^{\mathsf{T}} = \frac{r^2 |X|}{n} I_n$$

$$\langle \nabla \mathcal{E}(f_\alpha, L), H \rangle = -\alpha \sum_{x \in L \setminus \{0\}} H[x] e^{-\alpha \|x\|^2}.$$

$$= -\alpha \sum_{r > 0} e^{-\alpha r^2} \sum_{x \in L(r^2)} H[x].$$



$L(r^2) = \{x \in L : x \cdot x = r^2\}$ shell of L

$$\sum_{x \in L(r^2)} H[x] = \left\langle H, \sum_{x \in L(r^2)} x x^{\mathsf{T}} \right\rangle = \frac{r^2 |X|}{n} \mathrm{Tr}(H) = 0$$

$\Longrightarrow$ If every shell of L is spherical 2-design, then L is critical.

Spherical designs and eigenvalues of Hessian

Coulangeon, 2006

Similarly, if every shell of L is spherical 4-design, then

$$\nabla^2 \mathcal{E}(f_\alpha, L)[H] = \frac{\mathrm{Tr}\ H^2}{n(n+2)} \sum_{r>0} |L(r^2)| \alpha r^2 \left(\alpha r^2 - (n/2+1)\right) e^{-\alpha r^2}.$$

$\Longrightarrow$ all eigenvalues of $\nabla^2 \mathcal{E}(f_\alpha, L)$ coincide

Tool from analysis: Fourier transform

Fourier transform $\widehat{f}(u) = \int f(x) e^{-2\pi i x \cdot u} \, dx$

Eigenfunctions of Fourier transform

$$\int S_k(x) e^{-\pi |x|^2} e^{-2\pi i x \cdot u} \, dx = i^k S_k(u) e^{-\pi |u|^2}$$

$S_k(x)$ homogeneous harmonic polynomial of degree k

Poisson summation formula

$$\sum_{x \in L} f(x + v) = \frac{1}{\mathrm{vol}(L)} \sum_{u \in L^*} \widehat{f}(u) e^{2\pi i u \cdot y}$$

$L^* = \{u \in \mathbb{R}^n : x \cdot u \in \mathbb{Z} \text{ for all } v \in L\}$ **dual lattice**

Even unimodular lattices I

Probably the nicest lattices: $L^* = L$ and $x \cdot x \in 2\mathbb{Z}$ for all $x \in L$.

Venkov (1980)

Analyse these lattices with theta series with spherical coefficients

$$\Theta_{L,p}(\tau) = \sum_{x \in L} p(x) e^{\pi i \tau \|x\|^2} = \sum_{x \in L} p(x) q^{\frac{1}{2}\|x\|^2},$$

p harmonic polynomial

τ lies in the upper half plane $\{z \in \mathbb{C} : \Im(z) > 0\}$

$q = e^{2\pi i \tau}$

Even unimodular lattices 2

Poisson summation formula $\Longrightarrow$

Even unimodular lattices exist only when n is divisible by 8

$\Theta_{L,p}$ is a modular form of weight $n/2+k$, $k = \deg p$.

$\Theta_{L,p} \in \mathbb{C}[E_4, E_6]$

$$E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + \cdots ,$$

$$E_6(\tau) = 1 - 504q - 16632q^2 - 122976q^3 - \cdots ,$$

(normalized) Eisenstein series

Even unimodular lattices 3

For every fixed n there are only finitely many even unimodular lattices.

Completely classified for $n = 8, 16, 24$

$n = 8$ Mordell (1938) only E_8 root lattice.

$n = 16$ Witt (1941) D_{16}^+ and $E_8 \perp E_8$

$n = 24$ Niemeier (1973) Apart from the universally optimal Leech lattice Λ_{24} there are 23 further even unimodular lattices.

(classified by their root sublattices)

$n = 32$ ≥ 80 million

Known results

Theta series with spherical coefficients $\Longrightarrow$

All shells of even unimodular lattices for $n = 8, 16, 24$ are spherical 2-designs

All shells of $L = E_8, \Lambda_{24}$ are spherical 4-designs

Sarnak, Strömbergsson (2006): $\nabla^2 \mathcal{E}(f_\alpha, L)[H] > 0$ for all $\alpha > 0$

Need: Modified ∇^2-computation when shells are not spherical 4-designs.

Our modification

Theorem. Let L be an even unimodular lattice in dimension $n \leq 32$. Let

$$\Theta_L(\tau) = \sum_{m=0}^{\infty} a_m q^m \quad \text{with } a_m = |L(2m)|$$

be the theta series of L and let $\sum_{m=1}^{\infty} b_m q^m$ be the cusp form of weight $n/2+4$ with $b_1 = 1$. Then all the eigenvalues of the Hessian $\nabla^2 \mathcal{E}(f_\alpha, L)$ are given by

$$\begin{aligned} &\frac{1}{n(n+2)} \sum_{m=1}^{\infty} \left(b_m \frac{\alpha^2}{2} (\lambda n(n+2) - 8a_1) \right) e^{-2\alpha m} \\ &+ \frac{1}{n(n+2)} \sum_{m=1}^{\infty} \left(a_m 2\alpha m \left(2\alpha m - (n/2+1)\right)\right) e^{-2\alpha m}, \end{aligned} \tag{1}$$

where λ is an eigenvalue of the quadratic form

$$Q[H] = \sum_{x \in L(2)} H[x]^2$$

Eigenvalues

Theorem 4.1. Let R be an irreducible root system of type A, D, or E. The quadratic form $Q[H] = \sum_{x \in R} H[x]^2$ has the following eigenvalues:

Root system	Eigenvalue	Multiplicity
$A_n,\ n \geq 1$	$4h = 4(n+1)$	1
	$2(n+1)$	n, for $n \geq 2$
	4	$n(n-1)/2 - 1$, for $n \geq 2$
$D_n,\ n \geq 4$	$4h = 8(n-1)$	1
	$4(n-2)$	$n-1$
	8	$n(n-1)/2$
E_6	$4h = 48$	1
	12	20
E_7	$4h = 72$	1
	16	27
E_8	$4h = 120$	1
	24	35

Results for n = 16

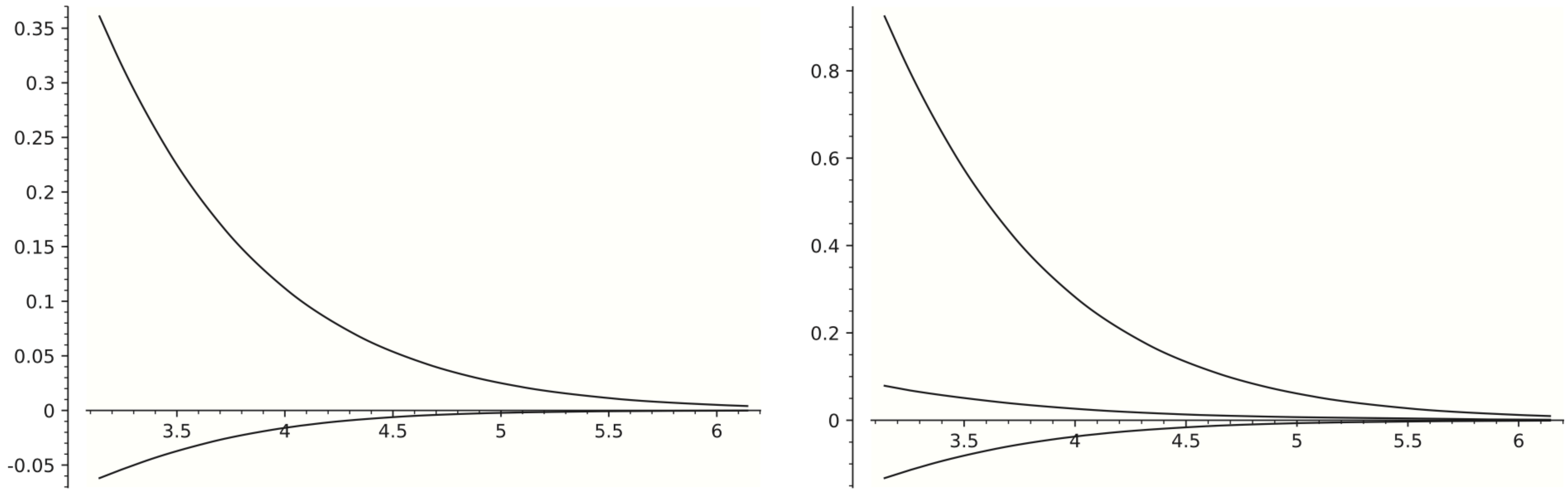


Fig. 1. The eigenvalues of the Hessian for D_{16}^{+} (two different eigenvalues, left) and $E_8 \perp E_8$ (three different eigenvalues, right) depending on the parameter α.

Results for n = 24

For large values of α: Only the Niemeier lattices with irreducible root systems, namely A_{24} and D_{24}, are local minima for f_α-potential energy. All other Niemeier lattices are saddle points for f_α-potential energy for α large enough.

There are no local maxima among the Niemeier lattices.

Some results for n = 32

King (2002): There are at least ten million even unimodular lattices without roots ($L(2) = \emptyset$) in dimension 32.

They all have the same theta series:

$$\begin{aligned}\Theta_L(\tau) &= E_4^4(\tau) - 960E_4(\tau)\Delta(\tau) \\ &= 1 + 146880q^2 + 64757760q^3 + 4844836800q^4 + 137695887360q^5 \\ &\qquad + 2121555283200q^6 + 21421110804480q^7 \\ &\qquad + 158757684004800q^8 + \cdots\end{aligned}$$

All shells of L form spherical 4-designs.

Eigenvalues of the Hessian

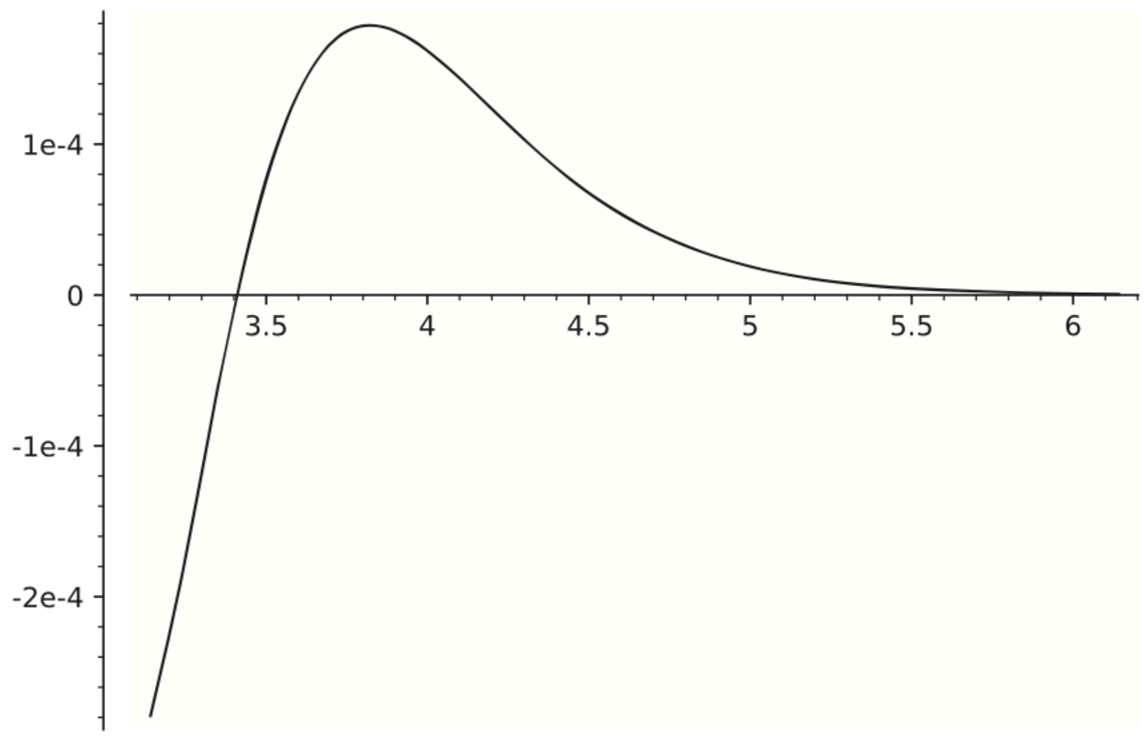


Fig. 2. The eigenvalue of the Hessian for even unimodular lattices in dimension 32 without roots depending on the parameter α.

$\Longrightarrow$ We found local maxima!