Round and Bipartize for Vertex Cover Approximation

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Vertex Cover Problem

Input: Graph $\mathcal{G} = (V, E)$ with weights $w : V \mapsto \mathbb{R}_+$

Goal: Find subset of vertices $U \subset V$ of minimum weight covering all the edges of the graph, i.e:

$$\min \left\{ w(U) \;\middle|\; U \subset V,\ |U \cap \{u, v\}| \geq 1 \quad \forall (u, v) \in E \right\}.$$

Integer Programming Formulation:

$$\begin{aligned} \min \sum_{v \in V} w_v x_v & \\ x_u + x_v \geq 1 & \quad \forall (u, v) \in E \\ x_v \in \{0, 1\} & \quad \forall v \in V \end{aligned}$$

Approximation Algorithms

Definition: Approximation Algorithm

An efficient algorithm for a minimization problem is a ϕ-approximation if it returns a solution U such that $w(U) \leq \phi\, w(\mathrm{OPT})$

Vertex Cover

- NP-Hard
- NP-Hard to approximate within a factor of 1.36 [Dinur, Safra]
- NP-Hard to approximate within $2 - \epsilon$ for any $\epsilon > 0$ under the unique games conjecture [Khot, Regev]
- Admits an easy 2-approximation using linear programming

Vertex Cover

Linear Programming Relaxation $P(\mathcal{G})$**:**

$$\min \sum_{v \in V} w_v x_v$$
$$x_u + x_v \geq 1 \quad \forall (u,v) \in E$$
$$x_v \geq 0 \quad \forall v \in V$$

Any extreme point solution $x^* \in [0,1]^V$ of $P(\mathcal{G})$ satisfies

- $x^* \in \{0,1\}^V$ for bipartite graphs $\mathcal{G}$
- $x^* \in \{0, \frac{1}{2}, 1\}^V$ for general graphs $\mathcal{G}$

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Bipartite $\mathcal{G}$: exact algorithm

Solve $P(\mathcal{G})$ to get $x^* \in \{0,1\}^V$
Return $U := \{v \in V \mid x_v^* = 1\}$

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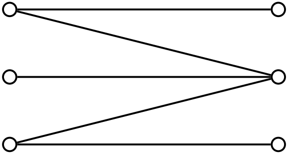
General $\mathcal{G}$: 2-approximation

Solve $P(\mathcal{G})$ to get $x^* \in \{0, \frac{1}{2}, 1\}^V$
Return $U := \{v \in V \mid x_v^* \geq \frac{1}{2}\}$

Vertex Cover: LP Relaxation



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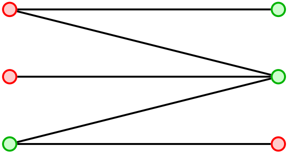

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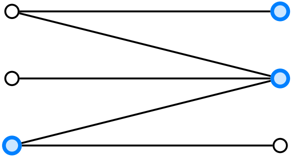

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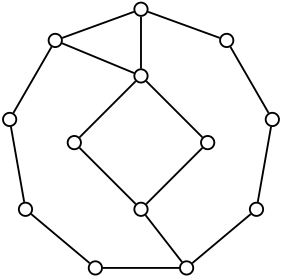

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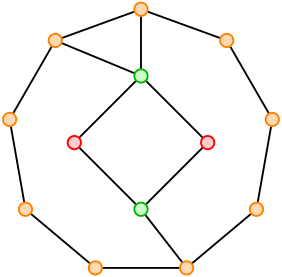

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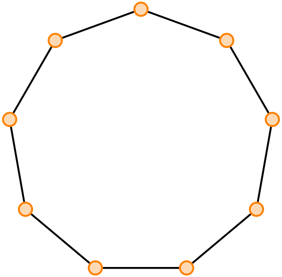

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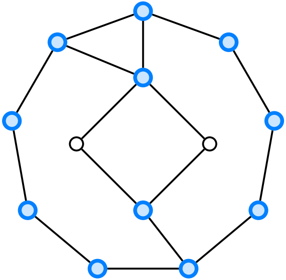

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Nemhauser-Trotter theorem

$\mathcal{G}_{1/2}$ is the subgraph induced by the half-integral nodes $V_{1/2}$.

Theorem [Nemhauser-Trotter]

Let $x^* \in \{0, \frac{1}{2}, 1\}^V$ be an optimal extreme point solution to $P(\mathcal{G})$. Then,

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Corollary

If $S \subset V_{1/2}$ is a ϕ-approximate solution for $\mathcal{G}_{1/2}$, then $S \cup V_1$ is a ϕ-approximate solution for $\mathcal{G}$.

Proof: $w(S) + w(V_1) \leq \phi\, w(\mathrm{OPT}(\mathcal{G}_{1/2})) + w(V_1) \leq \phi\, w(\mathrm{OPT}(\mathcal{G}))$

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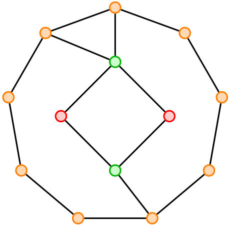

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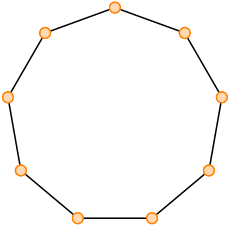

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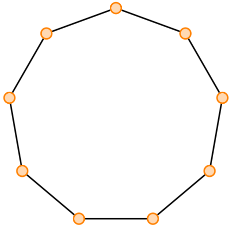

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Question

Possible to exploit some information about $\mathcal{G}_{1/2}$ for better approximation?

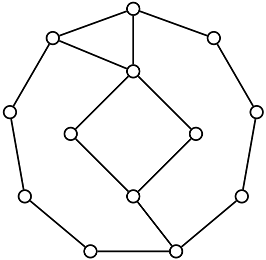
Our Set-Up

Algorithm: Additional Input

Suppose we have access to an odd cycle transversal S *of* $\mathcal{G}_{1/2}$*,*
i.e. $\mathcal{G}_{1/2} \setminus S$ *is a bipartite graph*

New Algorithm

- Solve the linear program $P(\mathcal{G})$ to get $V_0, V_1, V_{1/2}$
- Solve the integral linear program $P(\mathcal{G}_{1/2} \setminus S)$ to get $W \subset V_{1/2}$
- **Return:** $V_1 \cup S \cup W$



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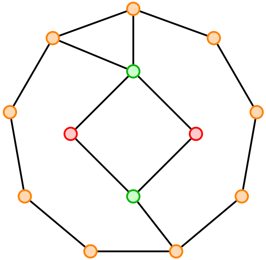

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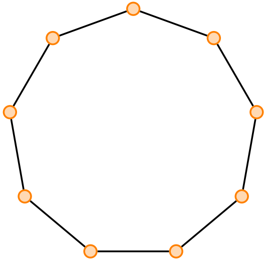

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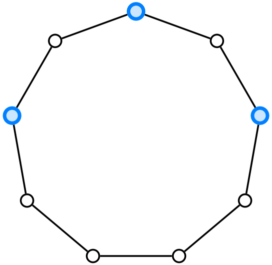

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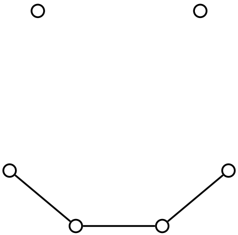

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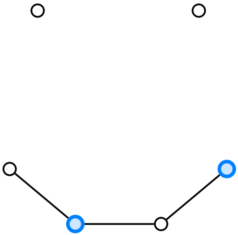

$$W = \mathrm{OPT}(\mathcal{G} \setminus S)$$

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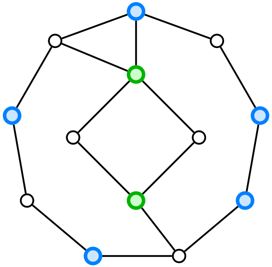

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- New/different view on a heavily studied problem

High-Level View

- Weight Space: for which weight functions $w : V \to \mathbb{R}$ is the solution $(1/2, \ldots, 1/2)$ optimal?
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Lemma

$(\frac{1}{2}, \dots, \frac{1}{2})$ *is optimal for* $P(\mathcal{G}) \iff \exists y \in \mathbb{R}_+^E$ *s.t.* $w_v = y(\delta(v)) \quad \forall v \in V$.

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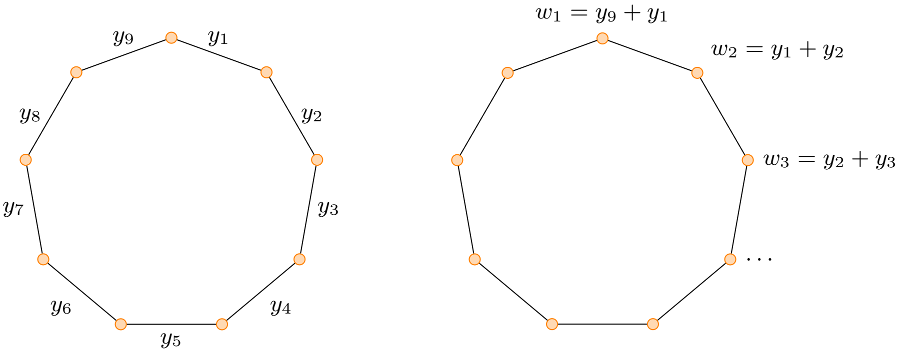

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Proof. By comp. slackness, a primal-dual pair (x, y) is optimal iff

$$x_v > 0 \implies y(\delta(v)) = w_v \quad \forall v \in V \tag{1}$$

$$y_e > 0 \implies x_u + x_v = 1 \quad \forall e = (u, v) \in E \tag{2}$$

$\implies$ Follows from condition (1)

$\impliedby$ The pair $(\frac{1}{2}, \ldots, \frac{1}{2}), y$ satisfy both conditions (1) and (2)

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$$\implies w(V) = 2y(E)$$

The approximation ratio $w(U)/w(\text{OPT})$ is invariant to scaling:

$$\implies \text{normalize } w(V) = 2 \text{ and } y(E) = 1$$

Weight Space

$$Q^W := \left\{ w \in \mathbb{R}_+^V \mid \exists y \in \mathbb{R}_+^E \text{ s.t. } y(E) = 1 \text{ and } w_v = y(\delta(v)) \quad \forall v \in V \right\}$$

Lower Bound on OPT

Lemma

Let $\mathcal{G} = (V, E)$ be a graph. For any $w \in Q^W$,

$$w(\mathrm{OPT}(\mathcal{G})) \geq 1$$

Proof. Since $w \in Q^W$, the solution $(\frac{1}{2}, \ldots, \frac{1}{2})$ is an optimal LP solution, by the previous slide. Its objective value (or cost) is

$$w(V)/2 = 1.$$

Since $\mathrm{OPT}(\mathcal{G})$ is a feasible LP solution, we get

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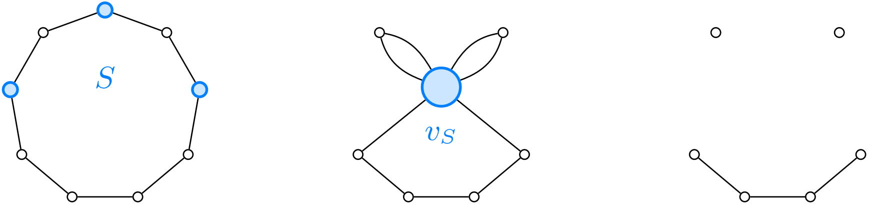
Stable Set to Bipartite



$\mathcal{G}$ $\qquad$ $\tilde{\mathcal{G}} = \mathcal{G}/S$ $\qquad$ $\mathcal{G}' = \mathcal{G} \setminus S$

Definition: parameter ρ

$2\rho - 1$ denotes the *odd girth* (length of the shortest odd cycle) of $\tilde{\mathcal{G}}$.
Hence, the range is $\rho \in [2, \infty]$

In the above example, $\rho = 3$, since the shortest odd cycle has length 5.

Stable Set to Bipartite

Algorithm/Approximation Ratio

Round on S and solve the integral linear program $P(\mathcal{G} \setminus S)$.

$$R(w) := \frac{w(S) + w(\mathrm{OPT}(\mathcal{G} \setminus S))}{w(\mathrm{OPT}(\mathcal{G}))}$$

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- Every weight function is assumed WLOG to satisfy $w \in Q^W$
- $\tilde{\mathcal{G}} = \mathcal{G}/S$ is the graph obtained after contracting S
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Theorem

Let $(\mathcal{G}, S)$ be the input, with S being a stable set. Then

$$R(w) \leq 1 + \frac{1}{\rho}$$

for every $w \in Q^W$. Equality holds for a convex subset $\mathcal{W} \subset Q^W$.

Tool to get improved bounds

$\mathcal{G}' = \mathcal{G} \setminus S$ is the bipartite graph obtained after removing S.

Definition

For a feasible vertex cover $U \subset V \setminus S$ of the bipartite graph $\mathcal{G}'$, we define

$$E_U := \left\{(u, v) \in E \;\middle|\; u \in U,\ v \in U \text{ or } \ u \in U,\ v \in S\right\}.$$

Covers $U_1, \ldots, U_k$ are *edge-separate* if $\{E_{U_1}, \ldots, E_{U_k}\}$ are pairwise disjoint.

Remark: Only need to cover $E(\mathcal{G}') = E \setminus \delta(S)$, but $E_U \subset E$

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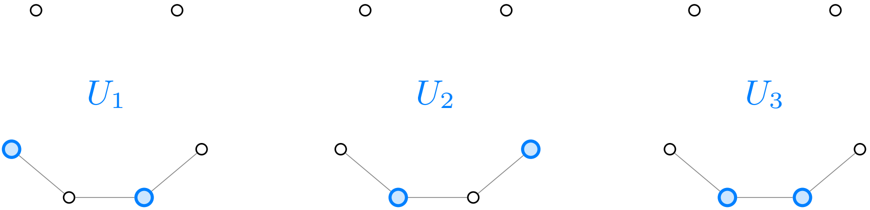

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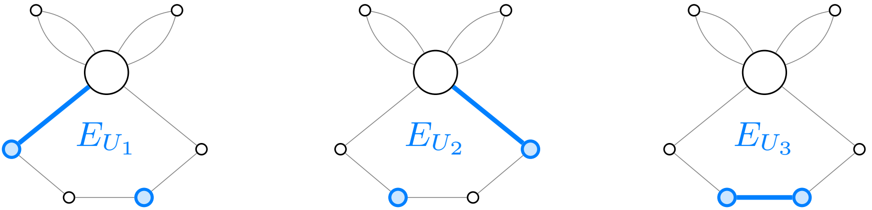

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Since $w_v = y(\delta(v))$, we can count the weight as

$$w(U) = y(E(\mathcal{G}')) + y(E_U)$$

because $E(\mathcal{G}')$ is counted at least once, by feasibility, with a surplus of E_U

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$$R(w) = \frac{w(S) + w(OPT(\mathcal{G}'))}{w(OPT(\mathcal{G}))} \leq w(S) + w(OPT(\mathcal{G}'))$$

$$\leq w(S) + \min_{i \in [k]} w(U_i) = y(\delta(S)) + y(E') + \min_{i \in [k]} y(E_{U_i})$$

$$= 1 + \min_{i \in [k]} y(E_{U_i}) \leq 1 + \frac{1}{k}$$

Constructing ρ covers

Lemma 2

There exists ρ edge-separate covers of $\mathcal{G}' = (A \cup B, E')$, where $2\rho - 1$ is the odd girth of the contracted graph $\tilde{\mathcal{G}} = \mathcal{G}/S$.

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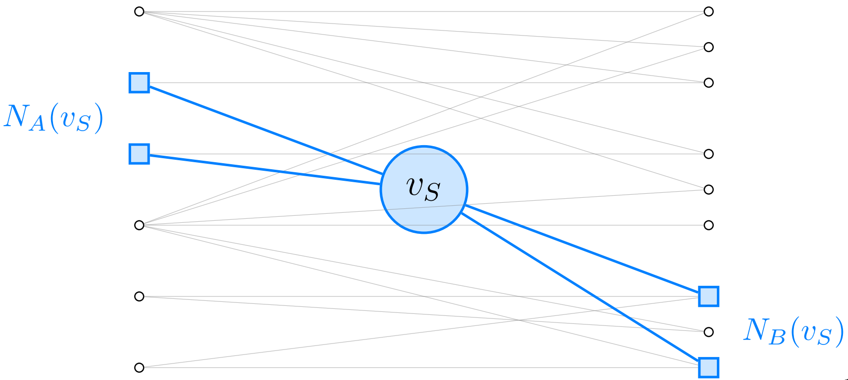

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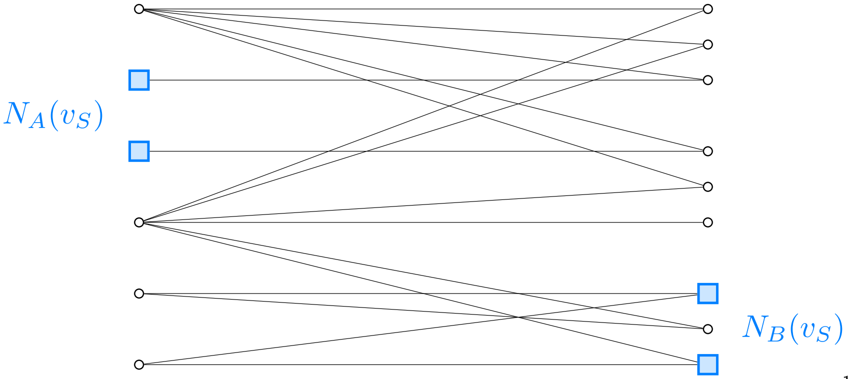

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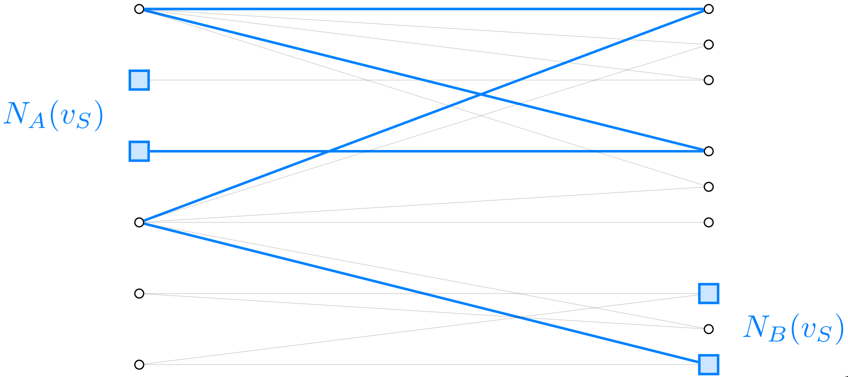

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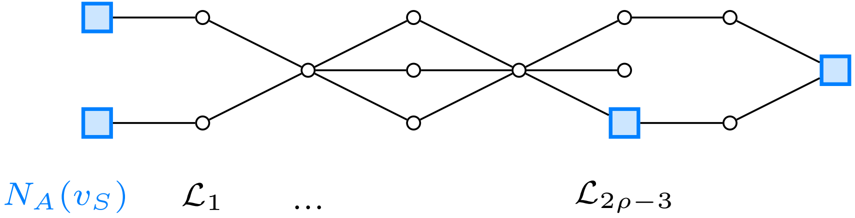
$$\mathcal{L}_i := \left\{ v \in A \cup B \mid d(N_A(v_S), v) = i \right\} \quad \forall i \geq 0$$



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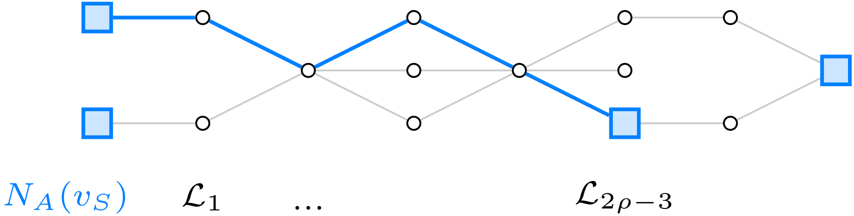

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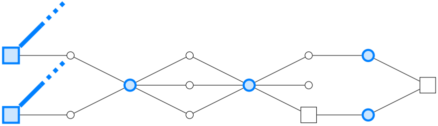
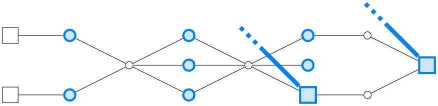
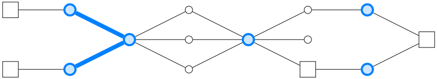


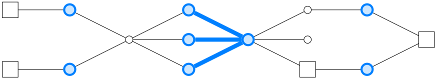
$E_{U_1} = E_A = \delta_A(v_S)$



$E_{U_2} = E_B = \delta_B(v_S)$



$E_{U_3} = E[\mathcal{L}_1, \mathcal{L}_2]$



$E_{U_4} = E[\mathcal{L}_3, \mathcal{L}_4]$

$1 + 1/\rho$ approximation

Theorem (Upper Bound)

Let $(\mathcal{G}, S)$ be the input, with S being a stable set. Then $R(w) \leq 1 + 1/\rho$ for every $w \in Q^W$, where $2\rho - 1$ is the odd girth of $\tilde{\mathcal{G}}$.

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Question.

Are there weight functions for which this bound is tight?

Tightness

Theorem

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Basic weight function corresponding to $C \in \mathcal{C}$

$$\forall C \in \mathcal{C}, \quad y^C : \tilde{E} \to \mathbb{R}_+ :$$

Set both dual edges incident to v_S to $1/\rho$ and then alternatingly set the dual edges to 0 and $1/\rho$ along the odd cycle on $\tilde{\mathcal{G}}$.

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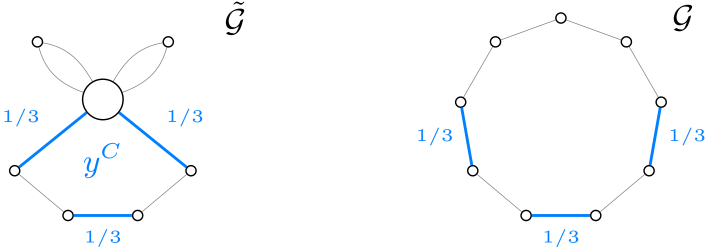

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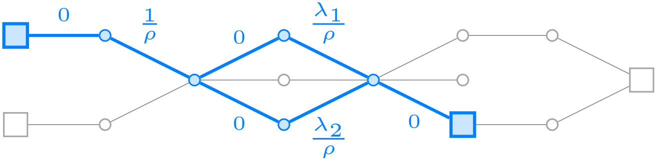

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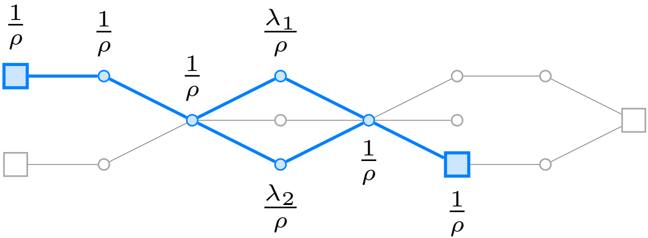

Integrality gap and fractional chromatic number

- Improved tight bounds on the integrality gap of $P(\mathcal{G})$ and fractional chromatic number for 3-colorable graphs.
- Exact formulas for $\tilde{\mathcal{G}}$.
- Proof based on the layer decomposition.

Theorem

$$\chi^f(\tilde{\mathcal{G}}) = 2 + \frac{1}{\rho - 1}, \quad IG(P(\tilde{\mathcal{G}})) = 1 + \frac{1}{2\rho - 1}$$

$\rightarrow$ highlights importance of the odd girth parameter ρ

High-Level View

- Weight Space: for which weight functions $w : V \to \mathbb{R}$ is the solution $(1/2, \ldots, 1/2)$ optimal?
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Arbitrary Set to Bipartite

Additional parameter

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Theorem

Let $(\mathcal{G}, S)$ be the input, where S is arbitrary odd cycle transversal. Then

$$R(w) \leq \left(1 + \frac{1}{\rho}\right)(1-\alpha) + 2\alpha$$

for every $w \in Q^W$. This bound is tight for any $\alpha \in [0,1]$ and $\rho \in [2, \infty]$.

- Interpolating "rounding curve" of the standard LP.
- Worst-case (standard 2-approximation): $\alpha = 1$ for $S = V_{1/2}$.
- Best-case (bipartite graphs): $\rho = \infty, \alpha = 0$.
- In-between (e.g. 3-colorable graphs): $\rho < \infty, \alpha = 0$.

Arbitrary Set to Bipartite

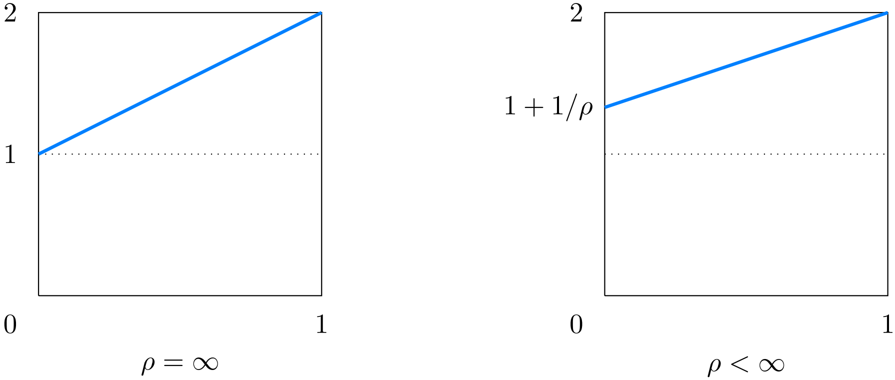

Figure: Plot of $R(w)$ with respect to $\alpha \in [0, 1]$

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Proof. Decompose the weight of S with respect to the dual variables:

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$$w(\mathrm{OPT}(\mathcal{G} \setminus S)) \leq \min_{i \in [\rho]} w(U_i) = y(E') + \min_{i \in [\rho]} y(E_{U_i}) \leq y(E') + \frac{1-\alpha}{\rho}$$

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$$\begin{aligned} R(w) &= \frac{w(S) + w(\mathrm{OPT}(\mathcal{G} \setminus S))}{w(\mathrm{OPT}(\mathcal{G}))} \le w(S) + w(\mathrm{OPT}(\mathcal{G} \setminus S)) \\ &\le 2\alpha + y(\delta(S)) + y(E') + \frac{1-\alpha}{\rho} \\ &= 1 + \alpha + \frac{1-\alpha}{\rho} = \left(1 + \frac{1}{\rho}\right)(1-\alpha) + 2\alpha. \end{aligned}$$

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Algorithmic Applications

Question.

How to find a good such set S algorithmically?

- Run an approximation/FPT algorithm for the min odd cycle transversal problem.
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- Find a k-coloring of $\mathcal{G}$ with stable sets $V_1 \cup \cdots \cup V_k$
- Order $w(V_1) \leq w(V_2) \cdots \leq w(V_k)$
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Can find an $\alpha := y(E[S]) \leq 1 - 4/k$

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Our worst-case bounds:

$$\rho = 2 \qquad \alpha = 1 - 4/n$$

Matching approximation

$$R(w) \leq \left(1 + \frac{1}{\rho}\right)(1-\alpha) + 2\alpha \; = \frac{3}{2}\,\frac{4}{n} + 2 - \frac{8}{n} = 2 - \frac{2}{n}$$

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- Beyond the worst-case analysis
- Interpolating the rounding curve of the standard LP from 1 to 2
- Motivation coming from algorithms with predictions
- Can compute improvement in the ratio once S is found/given
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Thanks!