A Primal-Dual Approximation Framework for Weighted Covering Problems

Britta Peis[1] Niklas Rieken[1] José Verschae[2] Andreas Wierz[1]

[1]RWTH Aachen University

[2]Pontificia Universidad Católica de Chile

Dutch Seminar, Nov 23, 2021

Covering and Packing Problems

- finite set of resources E with costs $c\colon E \to \mathbb{R}_+$
- demands $r\colon 2^E \to \mathbb{R}$ (non-decreasing)
- set-specific weights $a_{S,e} \in \mathbb{R}_+$ for all $S \subseteq E$ and $e \in S$.

Covering and Packing Problems

- finite set of resources E with costs $c\colon E \to \mathbb{R}_+$
- demands $r\colon 2^E \to \mathbb{R}$ (non-decreasing)
- set-specific weights $a_{S,e} \in \mathbb{R}_+$ for all $S \subseteq E$ and $e \in S$. Let $A = (a_{S,e}) \in \mathbb{R}_+^{2^E \times E}$.

Covering and Packing Problems

- finite set of resources E with costs $c\colon E \to \mathbb{R}_+$
- demands $r\colon 2^E \to \mathbb{R}$ (non-decreasing)
- set-specific weights $a_{S,e} \in \mathbb{R}_+$ for all $S \subseteq E$ and $e \in S$. Let $A = (a_{S,e}) \in \mathbb{R}_+^{2^E \times E}$.

We consider weighted integer covering problems of type

$$(\hat{P}) \qquad \min_{x \in \mathbb{Z}_+^E} \left\{ \sum_{e \in E} c(e) x_e \,\middle|\, \sum_{e \in S} a_{S,e} x_e \geq r(S), S \subseteq E \right\}$$

Covering and Packing Problems

- finite set of resources E with costs $c\colon E \to \mathbb{R}_+$
- demands $r\colon 2^E \to \mathbb{R}$ (non-decreasing)
- set-specific weights $a_{S,e} \in \mathbb{R}_+$ for all $S \subseteq E$ and $e \in S$. Let $A = (a_{S,e}) \in \mathbb{R}_+^{2^E \times E}$.

We consider weighted integer covering problems of type

$$(\hat{P}) \qquad \min \left\{ c^T x \mid Ax \geq r, x \in \mathbb{Z}_+^E \right\}$$

Covering and Packing Problems

- finite set of resources E with costs $c\colon E \to \mathbb{R}_+$
- demands $r\colon 2^E \to \mathbb{R}$ (non-decreasing)
- set-specific weights $a_{S,e} \in \mathbb{R}_+$ for all $S \subseteq E$ and $e \in S$. Let $A = (a_{S,e}) \in \mathbb{R}_+^{2^E \times E}$.

We consider weighted integer covering problems of type

$$(\hat{P}) \qquad \min\left\{c^T x \mid Ax \geq r, x \in \mathbb{Z}_+^E\right\}$$

Linear relaxation and it's dual

$$(P) \qquad \min_{x \geq 0}\{c^T x \mid Ax \geq r\} \qquad\qquad (D) \qquad \max_{y \geq 0}\{r^T y \mid A^T y \leq c\}$$

Covering and Packing Problems

- finite set of resources E with costs $c\colon E \to \mathbb{R}_+$
- demands $r\colon 2^E \to \mathbb{R}$ (non-decreasing)
- set-specific weights $a_{S,e} \in \mathbb{R}_+$ for all $S \subseteq E$ and $e \in S$. Let $A = (a_{S,e}) \in \mathbb{R}_+^{2^E \times E}$.

We consider weighted integer covering problems of type

$$(\hat{P}) \qquad \min\left\{c^T x \mid Ax \geq r, x \in \mathbb{Z}_+^E\right\}$$

Linear relaxation and it's dual

$$(P) \quad \min_{x \geq 0}\{c^T x \mid Ax \geq r\} \qquad\qquad (D) \quad \max_{y \geq 0}\{r^T y \mid A^T y \leq c\}$$

Question

Characterization of systems (A, r) for which greedy algorithms find feasible (integral) solutions of "good" performance guarantee for all c?

Covering and Packing Problems

- finite set of resources E with costs $c\colon E \to \mathbb{R}_+$
- demands $r\colon 2^E \to \mathbb{R}$ (non-decreasing)
- set-specific weights $a_{S,e} \in \mathbb{R}_+$ for all $S \subseteq E$ and $e \in S$. Let $A = (a_{S,e}) \in \mathbb{R}_+^{2^E \times E}$.

We consider weighted integer covering problems of type

$$(\hat{P}) \qquad \min \left\{ c^T x \mid Ax \geq r, x \in \mathbb{Z}_+^E \right\}$$

Linear relaxation and it's dual

$$(P) \quad \min_{x \geq 0} \{ c^T x \mid Ax \geq r \} \qquad\qquad (D) \quad \max_{y \geq 0} \{ r^T y \mid A^T y \leq c \}$$

Question

Characterization of systems (A, r) for which greedy algorithms find feasible (integral) solutions of "good" performance guarantee for all c?

- **Dual greedy:** *iteratively raise "largest" dual variable as far as possible.*
- **Primal greedy:** *iteratively select element giving the "best bang for the buck".*

Special case: $\{0, 1\}$-matrices

Polymatroid greedy algorithm

Consider special case where A is $\{0,1\}$-matrix with $a_{S,e} = 1$ iff $e \in S$.

$$(P)\ \min_{x \geq 0}\{c^T x \mid \sum_{e \in S} x_e \geq r(S)\ \forall S \subseteq E\} \qquad (D)\ \max_{y \geq 0}\{r^T y \mid \sum_{S:\ e \in S} y_S \leq c(e)\ \forall e \in E\}$$

Polymatroid greedy algorithm

Consider special case where A is $\{0,1\}$-matrix with $a_{S,e} = 1$ iff $e \in S$.

$$(P)\ \min_{x \geq 0} \{c^T x \mid \sum_{e \in S} x_e \geq r(S)\ \forall S \subseteq E\} \qquad (D)\ \max_{y \geq 0} \{r^T y \mid \sum_{S:\, e \in S} y_S \leq c(e)\ \forall e \in E\}$$

Function $r : 2^E \to \mathbb{R}$ is supermodular if

$$r(S \cup \{e\}) - r(S) \leq r(T \cup \{e\}) - r(T) \quad \forall S \subseteq T \subseteq E \setminus \{e\}, \forall e \in E$$

Polymatroid greedy algorithm

Consider special case where A is $\{0,1\}$-matrix with $a_{S,e} = 1$ iff $e \in S$.

$$(P)\ \min_{x\geq 0}\{c^T x \mid \sum_{e\in S} x_e \geq r(S)\ \forall S \subseteq E\} \qquad (D)\ \max_{y\geq 0}\{r^T y \mid \sum_{S:\, e\in S} y_S \leq c(e)\ \forall e \in E\}$$

Function $r : 2^E \to \mathbb{R}$ is supermodular if

$$r(S \cup \{e\}) - r(S) \leq r(T \cup \{e\}) - r(T) \quad \forall S \subseteq T \subseteq E \setminus \{e\}, \forall e \in E$$

"The incentives for joining a coalition increase as the coalition grows" [Shapley '71]

Polymatroid greedy algorithm

Consider special case where A is $\{0,1\}$-matrix with $a_{S,e} = 1$ iff $e \in S$.

$$(P)\ \min_{x \geq 0}\{c^T x \mid \sum_{e \in S} x_e \geq r(S)\ \forall S \subseteq E\} \qquad (D)\ \max_{y \geq 0}\{r^T y \mid \sum_{S:\, e \in S} y_S \leq c(e)\ \forall e \in E\}$$

Function $r : 2^E \to \mathbb{R}$ is supermodular if

$$r(S \cup \{e\}) - r(S) \leq r(T \cup \{e\}) - r(T) \quad \forall S \subseteq T \subseteq E \setminus \{e\}, \forall e \in E$$

"The incentives for joining a coalition increase as the coalition grows" [Shapley '71]

Theorem (Edmonds' 70s)

If r is supermodular, optimal solutions x^* and y^* for (P) and (D) can be computed with the polymatroid greedy algorithm.

Polymatroid greedy algorithm

Consider special case where A is $\{0,1\}$-matrix with $a_{S,e} = 1$ iff $e \in S$.

$$(P)\ \min_{x \geq 0}\{c^T x \mid \sum_{e \in S} x_e \geq r(S)\ \forall S \subseteq E\} \qquad (D)\ \max_{y \geq 0}\{r^T y \mid \sum_{S:\, e \in S} y_S \leq c(e)\ \forall e \in E\}$$

Function $r : 2^E \to \mathbb{R}$ is supermodular if

$$r(S \cup \{e\}) - r(S) \leq r(T \cup \{e\}) - r(T) \quad \forall S \subseteq T \subseteq E \setminus \{e\}, \forall e \in E$$

"The incentives for joining a coalition increase as the coalition grows" [Shapley '71]

Theorem (Edmonds' 70s)

If r is supermodular, optimal solutions x^* and y^* for (P) and (D) can be computed with the polymatroid greedy algorithm. Moreover, x^* and y^* are integral whenever r and c are integral.

Proof: later.

Application: Core computation in cooperative convex games

A cooperative (benefit) game is a tuple (N, v) consisting of

- a finite set $N = \{1, \ldots, n\}$ of agents
- a benefit function $v : 2^N \to \mathbb{R}_+$ (non-decreasing).

Application: Core computation in cooperative convex games

A cooperative (benefit) game is a tuple (N, v) consisting of

- a finite set $N = \{1, \ldots, n\}$ of agents
- a benefit function $v : 2^N \to \mathbb{R}_+$ (non-decreasing).

Interpretation: For each coalition $S \subseteq N$, the value $v(S)$ denotes the joint benefit coalition S can achieve if only the agents in S cooperate.

Application: Core computation in cooperative convex games

A cooperative (benefit) game is a tuple (N, v) consisting of

- a finite set $N = \{1, \dots, n\}$ of agents
- a benefit function $v : 2^N \to \mathbb{R}_+$ (non-decreasing).

Interpretation: For each coalition $S \subseteq N$, the value $v(S)$ denotes the joint benefit coalition S can achieve if only the agents in S cooperate.

Definition

The core of cooperative game (N, v) consists of all allocations in

$$\text{Core}(N, v) := \{x \in \mathbb{R}^N_+ \mid \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \ \forall S \subseteq N\}.$$

Application: Core computation in cooperative convex games

A cooperative (benefit) game is a tuple (N, v) consisting of

- a finite set $N = \{1, \dots, n\}$ of agents
- a benefit function $v : 2^N \to \mathbb{R}_+$ (non-decreasing).

Interpretation: For each coalition $S \subseteq N$, the value $v(S)$ denotes the joint benefit coalition S can achieve if only the agents in S cooperate.

Definition

The core of cooperative game (N, v) consists of all allocations in

$$\text{Core}(N, v) := \{x \in \mathbb{R}_+^N \mid \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \ \forall S \subseteq N\}.$$

Theorem (Shapley '71)

The cooperative game (N, v) is convex iff v is supermodular. Convex cooperative games are exactly the games such that $\forall \pi : N \to N$ the associated marginal-benefit allocation x^π lies in Core(N, v).

Polymatroid greedy: the algorithm

$$(P)\ \min_{x\geq 0}\{c^T x \mid \sum_{e\in S} x_e \geq r(S)\ \forall S\subseteq E\} \qquad (D)\ \max_{y\geq 0}\{r^T y \mid \sum_{S:\, e\in S} y_S \leq c(e)\ \forall e\in E\}$$

Polymatroid greedy: the algorithm

$$(P)\ \min_{x\geq 0}\{c^T x \mid \sum_{e\in S} x_e \geq r(S)\ \forall S\subseteq E\} \qquad (D)\ \max_{y\geq 0}\{r^T y \mid \sum_{S:\, e\in S} y_S \leq c(e)\ \forall e\in E\}$$

Dual Greedy:

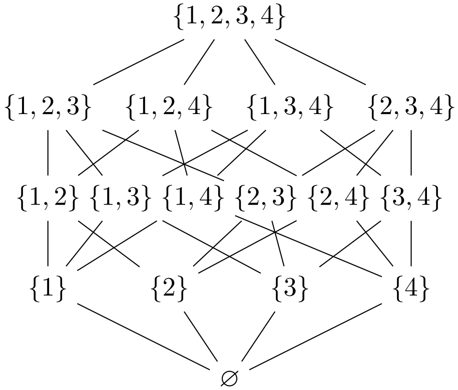
$y^* := \vec{0}, S := E$
while $r(S) > 0$ **do**
  Raise $y^*(S)$ until some $e^* \in S$ gets tight
  $S := S \setminus \{e^*\}$
return y^*

Polymatroid greedy: the algorithm

$$(P)\ \min_{x \geq 0}\{c^T x \mid \sum_{e \in S} x_e \geq r(S)\ \forall S \subseteq E\} \qquad (D)\ \max_{y \geq 0}\{r^T y \mid \sum_{S:\, e \in S} y_S \leq c(e)\ \forall e \in E\}$$

Dual Greedy:

$y^* := \vec{0}, S := E$
while $r(S) > 0$ **do**
  Raise $y^*(S)$ until some $e^* \in S$ gets tight
  $S := S \setminus \{e^*\}$
return y^*



Polymatroid greedy: the algorithm

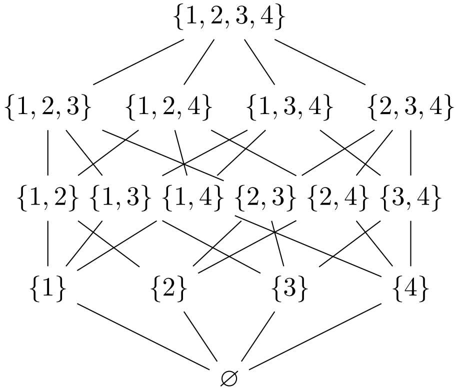
$$(P)\ \min_{x\geq 0}\{c^T x \mid \sum_{e\in S} x_e \geq r(S)\ \forall S\subseteq E\} \qquad (D)\ \max_{y\geq 0}\{r^T y \mid \sum_{S:\, e\in S} y_S \leq c(e)\ \forall e\in E\}$$

Dual Greedy:

$y^* := \vec{0}, S := E$
while $r(S) > 0$ **do**
  Raise $y^*(S)$ until some $e^* \in S$ gets tight
  $S := S \setminus \{e^*\}$
return y^*

Note: Greedy constructs chain $S_k \subset S_{k-1} \subset \ldots \subset S_1 = E$ with

$$e_i = S_i \backslash S_{i+1} \text{ for } e_i \in E^* = \{e_k, \ldots, e_1\}.$$



Polymatroid greedy: the algorithm

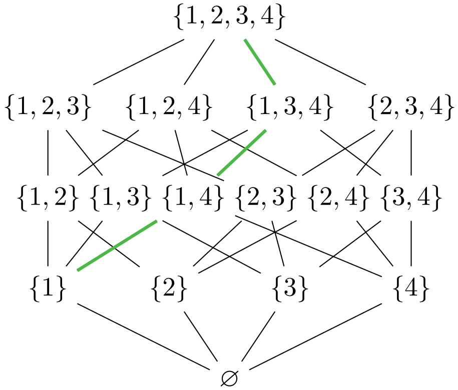
$$(P)\ \min_{x\geq 0}\{c^T x \mid \sum_{e\in S} x_e \geq r(S)\ \forall S\subseteq E\} \qquad (D)\ \max_{y\geq 0}\{r^T y \mid \sum_{S:\, e\in S} y_S \leq c(e)\ \forall e\in E\}$$

Dual Greedy:

$y^* := \vec{0}, S := E$
while $r(S) > 0$ **do**
  Raise $y^*(S)$ until some $e^* \in S$ gets tight
  $S := S \setminus \{e^*\}$
return y^*

Note: Greedy constructs chain $S_k \subset S_{k-1} \subset \ldots \subset S_1 = E$ with

$$e_i = S_i \setminus S_{i+1} \text{ for } e_i \in E^* = \{e_k, \ldots, e_1\}.$$



Polymatroid greedy: the algorithm

$$(P)\ \min_{x \geq 0}\{c^T x \mid \sum_{e \in S} x_e \geq r(S)\ \forall S \subseteq E\} \qquad (D)\ \max_{y \geq 0}\{r^T y \mid \sum_{S:\, e \in S} y_S \leq c(e)\ \forall e \in E\}$$

Dual Greedy:

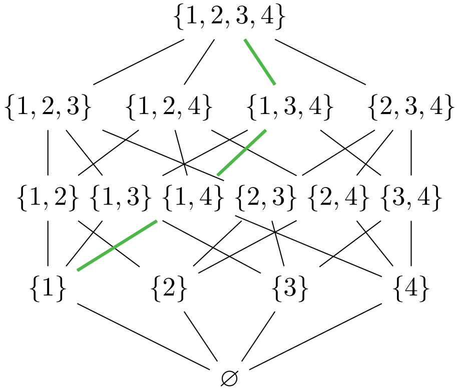
$y^* := \vec{0}, S := E$
while $r(S) > 0$ **do**
  Raise $y^*(S)$ until some $e^* \in S$ gets tight
  $S := S \setminus \{e^*\}$
return y^*

Note: Greedy constructs chain $S_k \subset S_{k-1} \subset \ldots \subset S_1 = E$ with

$$e_i = S_i \backslash S_{i+1} \text{ for } e_i \in E^* = \{e_k, \ldots, e_1\}.$$

Construct primal candidate: $x^* \in \mathbb{R}^E_+$ with non-zero values

$$x^*(e_i) = \begin{cases} r(S_k) & \text{if } i = k \\ r(S_i) - r(S_{i+1}) & \text{if } i = 1, \ldots, k-1 \end{cases}$$



Polymatroid greedy: the algorithm

$$(P)\ \min_{x\geq 0}\{c^T x \mid \sum_{e\in S} x_e \geq r(S)\ \forall S\subseteq E\} \qquad (D)\ \max_{y\geq 0}\{r^T y \mid \sum_{S:\, e\in S} y_S \leq c(e)\ \forall e\in E\}$$

Dual Greedy:

$y^* := \vec{0}, S := E$
while $r(S) > 0$ **do**
  Raise $y^*(S)$ until some $e^* \in S$ gets tight
  $S := S \setminus \{e^*\}$
return y^*

Lemma

y^* is feasible for (D), and

(i) $x^*(e) > 0$ *implies* $\sum_{S:e\in S} y^*(S) = c(e)$

(ii) $y^*(S) > 0$ *implies* $\sum_{e\in S} x^*(e) = r(S)$

Note: Greedy constructs chain $S_k \subset S_{k-1} \subset \ldots \subset S_1 = E$ with

$$e_i = S_i \setminus S_{i+1} \text{ for } e_i \in E^* = \{e_k, \ldots, e_1\}.$$

Construct primal candidate: $x^* \in \mathbb{R}^E_+$ with non-zero values

$$x^*(e_i) = \begin{cases} r(S_k) & \text{if } i = k \\ r(S_i) - r(S_{i+1}) & \text{if } i = 1, \ldots, k-1 \end{cases}$$

Polymatroid greedy algorithm (cont.)

Lemma

If r is non-decreasing and supermodular, then x^* is feasible for (P).

Polymatroid greedy algorithm (cont.)

Lemma

If r is non-decreasing and supermodular, then x^* is feasible for (P).

Proof:

- $x^*(e_i) = r(S_i) - r(S_{i+1}) \geq 0$ by mononiticy of r.

Polymatroid greedy algorithm (cont.)

Lemma

If r is non-decreasing and supermodular, then x^* is feasible for (P).

Proof:

- $x^*(e_i) = r(S_i) - r(S_{i+1}) \geq 0$ by mononiticy of r.
- Remains to show:

$$\text{gap}(T) := \sum_{e \in T} x^*(e) - r(T) \geq 0 \quad \forall T \subseteq E.$$

Polymatroid greedy algorithm (cont.)

Lemma

If r is non-decreasing and supermodular, then x^* is feasible for (P).

Proof:

- $x^*(e_i) = r(S_i) - r(S_{i+1}) \geq 0$ by mononiticy of r.
- Remains to show:

$$\text{gap}(T) := \sum_{e \in T} x^*(e) - r(T) \geq 0 \quad \forall T \subseteq E.$$

- Let T be minimal with gap$(T) < 0$. Select i maximal with $T \subseteq S_i$.

Polymatroid greedy algorithm (cont.)

Lemma

If r is non-decreasing and supermodular, then x^* is feasible for (P).

Proof:

- $x^*(e_i) = r(S_i) - r(S_{i+1}) \geq 0$ by mononiticy of r.
- Remains to show:

$$\text{gap}(T) := \sum_{e \in T} x^*(e) - r(T) \geq 0 \quad \forall T \subseteq E.$$

- Let T be minimal with $\text{gap}(T) < 0$. Select i maximal with $T \subseteq S_i$.
- Note that gap is submodular and $\text{gap}(S_i) = 0$ for all $i \in [k]$. Derive contradiction

$$\text{gap}(T) \geq \text{gap}(T \cup S_{i+1}) + \text{gap}(T \cap S_{i+1}) - \text{gap}(S_{i+1}) \geq 0.$$

Complementary Slackness

Corollary

If r is monotone and supermodular, then x^* and y^* are feasible for (P) and (D) and satisfy the complementary slackness conditions

(i) $x^*(e) > 0$ *implies* $\sum_{S:e\in S} y^*(S) = c(e)$

(ii) $y^*(S) > 0$ *implies* $\sum_{e\in S} x^*(e) = r(S)$

Complementary Slackness

Corollary

If r is monotone and supermodular, then x^* and y^* are feasible for (P) and (D) and satisfy the complementary slackness conditions

(i) $x^*(e) > 0$ *implies* $\sum_{S:e\in S} y^*(S) = c(e)$

(ii) $y^*(S) > 0$ *implies* $\sum_{e\in S} x^*(e) = r(S)$

Thus, by Complementary Slackness, x^* and y^* are optimal.

Complementary Slackness

Corollary

If r is monotone and supermodular, then x^* and y^* are feasible for (P) and (D) and satisfy the complementary slackness conditions

(i) $x^*(e) > 0$ *implies* $\sum_{S:e\in S} y^*(S) = c(e)$

(ii) $y^*(S) > 0$ *implies* $\sum_{e\in S} x^*(e) = r(S)$

Thus, by Complementary Slackness, x^* and y^* are optimal.

Proof: By weak LP-duality: for feasible primal and dual solutions x and y holds

$$r^T y \leq (Ax)^T y = x^T(A^T y) \leq x^T c = c^T x.$$

Complementary Slackness

Corollary

If r is monotone and supermodular, then x^* and y^* are feasible for (P) and (D) and satisfy the complementary slackness conditions

(i) $x^*(e) > 0$ *implies* $\sum_{S:e\in S} y^*(S) = c(e)$

(ii) $y^*(S) > 0$ *implies* $\sum_{e\in S} x^*(e) = r(S)$

Thus, by Complementary Slackness, x^* and y^* are optimal.

Proof: By weak LP-duality: for feasible primal and dual solutions x and y holds

$$r^T y \leq (Ax)^T y = x^T(A^T y) \leq x^T c = c^T x.$$

Conditions (i) and (ii) imply optimality, since objective function values coincide:

$$c^T x^* = \sum_{e\in E} c(e)x^*(e) \overset{(i)}{=} \sum_{e\in E}\sum_{S:e\in S} y^*(S)x^*(e) = \sum_{S\subseteq E} y^*(S) \sum_{e\in S} x^*(e)$$

$$\overset{(ii)}{=} \sum_{S\subseteq E} y^*(S)r(S) = r^T y^*$$

General matrices $A \in \mathbb{R}_+^{2^E \times E}$

Covering and Packing Problems

- finite set of resources E with costs $c\colon E \to \mathbb{R}_+$
- demands $r\colon 2^E \to \mathbb{R}$ (non-decreasing)
- set-specific weights $a_{S,e} \in \mathbb{R}_+$ for all $S \subseteq E$ and $e \in S$.

Covering and Packing Problems

- finite set of resources E with costs $c\colon E \to \mathbb{R}_+$
- demands $r\colon 2^E \to \mathbb{R}$ (non-decreasing)
- set-specific weights $a_{S,e} \in \mathbb{R}_+$ for all $S \subseteq E$ and $e \in S$. Let $A = (a_{S,e}) \in \mathbb{R}_+^{2^E \times E}$.

We consider weighted integer covering problems of type

$$(\hat{P}) \qquad \min\left\{c^T x \mid Ax \geq r, x \in \mathbb{Z}_+^E\right\}$$

Covering and Packing Problems

- finite set of resources E with costs $c\colon E \to \mathbb{R}_+$
- demands $r\colon 2^E \to \mathbb{R}$ (non-decreasing)
- set-specific weights $a_{S,e} \in \mathbb{R}_+$ for all $S \subseteq E$ and $e \in S$. Let $A = (a_{S,e}) \in \mathbb{R}_+^{2^E \times E}$.

We consider weighted integer covering problems of type

$$(\hat{P}) \qquad \min \left\{ c^T x \mid Ax \geq r, x \in \mathbb{Z}_+^E \right\}$$

Linear relaxation and it's dual

$$(P) \qquad \min_{x \geq 0} \{ c^T x \mid Ax \geq r \} \qquad\qquad (D) \qquad \max_{y \geq 0} \{ r^T y \mid A^T y \leq c \}$$

Covering and Packing Problems

- finite set of resources E with costs $c\colon E \to \mathbb{R}_+$
- demands $r\colon 2^E \to \mathbb{R}$ (non-decreasing)
- set-specific weights $a_{S,e} \in \mathbb{R}_+$ for all $S \subseteq E$ and $e \in S$. Let $A = (a_{S,e}) \in \mathbb{R}_+^{2^E \times E}$.

We consider weighted integer covering problems of type

$$(\hat{P}) \qquad \min\left\{c^T x \mid Ax \geq r, x \in \mathbb{Z}_+^E\right\}$$

Linear relaxation and it's dual

$$(P) \qquad \min_{x\geq 0}\{c^T x \mid Ax \geq r\} \qquad\qquad (D) \qquad \max_{y\geq 0}\{r^T y \mid A^T y \leq c\}$$

Question

Characterization of systems (A, r) for which greedy algorithms find feasible (integral) solutions of "good" performance guarantee for all c?

- **Dual greedy:** *iteratively raise "largest" dual variable as far as possible.*
- **Primal greedy:** *iteratively select element giving the "best bang for the buck".*

Dual Greedy Algorithm

So far: Greedy algorithm works optimally for all c iff A is $\{0,1\}$-incidence matrix of 2^E and r is monotone increasing and supermodular.

So far: Greedy algorithm works optimally for all c iff A is $\{0,1\}$-incidence matrix of 2^E and r is monotone increasing and supermodular.

$$\min_{x \geq 0}\{c^T x \mid \sum_{e \in S} a_{S,e} x_e \geq r(S)\ \forall S \subseteq E\} \qquad \max_{y \geq 0}\{r^T y \mid \sum_{S:\, e \in S} a_{S,e} y_S \leq c(e)\ \forall e \in E\}$$

So far: Greedy algorithm works optimally for all c iff A is $\{0,1\}$-incidence matrix of 2^E and r is monotone increasing and supermodular.

$$\min_{x \geq 0}\{c^T x \mid \sum_{e \in S} a_{S,e} x_e \geq r(S) \ \forall S \subseteq E\} \qquad \max_{y \geq 0}\{r^T y \mid \sum_{S:\ e \in S} a_{S,e} y_S \leq c(e) \ \forall e \in E\}$$

Assumption: $S \subseteq T$ implies $r(S) \leq r(T)$ and $a_{S,e} \leq a_{T,e}$ for all $e \in E$.

So far: Greedy algorithm works optimally for all c iff A is $\{0,1\}$-incidence matrix of 2^E and r is monotone increasing and supermodular.

$$\min_{x\geq 0}\{c^T x \mid \sum_{e\in S} a_{S,e} x_e \geq r(S)\ \forall S\subseteq E\} \qquad \max_{y\geq 0}\{r^T y \mid \sum_{S:\ e\in S} a_{S,e} y_S \leq c(e)\ \forall e\in E\}$$

Assumption: $S\subseteq T$ implies $r(S)\leq r(T)$ and $a_{S,e}\leq a_{T,e}$ for all $e\in E$.

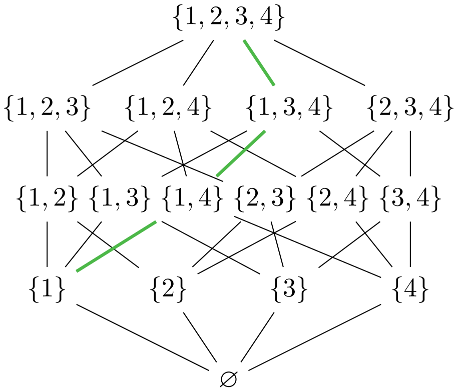
So far: Greedy algorithm works optimally for all c iff A is $\{0,1\}$-incidence matrix of 2^E and r is monotone increasing and supermodular.

$$\min_{x\geq 0}\{c^T x \mid \sum_{e\in S} a_{S,e}x_e \geq r(S)\ \forall S\subseteq E\} \qquad \max_{y\geq 0}\{r^T y \mid \sum_{S:\, e\in S} a_{S,e}y_S \leq c(e)\ \forall e\in E\}$$

Assumption: $S\subseteq T$ implies $r(S)\leq r(T)$ and $a_{S,e}\leq a_{T,e}$ for all $e\in E$.

Dual Greedy:

$y^* := \vec{0}, S := E$
while $r(S) > 0$ **do**
  Raise $y^*(S)$ until some $e^* \in S$ gets tight
  $S := S \setminus \{e^*\}$
return y^*



Note: Greedy constructs chain $S_k \subset S_{k-1} \subset \ldots \subset S_1 = E$ with $e_i = S_i \setminus S_{i+1}$ for $e_i \in E^* = \{e_k, \ldots, e_1\}$.

Dual Greedy

$$\min_{x\geq 0}\{c^T x \mid \sum_{e\in S} a_{S,e}x_e \geq r(S)\ \forall S\subseteq E\} \qquad \max_{y\geq 0}\{r^T y \mid \sum_{S:\, e\in S} a_{S,e}y_S \leq c(e)\ \forall e\in E\}$$

Dual Greedy:

$y^* := \vec{0}, S := E$
while $r(S) > 0$ *and* $S \neq \varnothing$ **do**
  Raise $y^*(S)$ until some $e^* \in S$ gets tight
  $S := S \setminus \{e^*\}$
return y^*

Dual Greedy

$$\min_{x\geq 0}\{c^T x \mid \sum_{e\in S} a_{S,e}x_e \geq r(S)\ \forall S\subseteq E\} \qquad \max_{y\geq 0}\{r^T y \mid \sum_{S:\, e\in S} a_{S,e}y_S \leq c(e)\ \forall e\in E\}$$

Dual Greedy:

$y^* := \vec{0}, S := E$
while $r(S) > 0$ *and* $S \neq \varnothing$ **do**
  Raise $y^*(S)$ until some $e^* \in S$ gets tight
  $S := S \setminus \{e^*\}$
return y^*

Construct primal candidate: Based on chain $S_k \subset S_{k-1} \subset \ldots \subset S_1 = E$ and bottleneck elements $E^* = \{e_k, \ldots, e_1\}$ construct $x^* \in \mathbb{R}^E_+$ with non-zero values

$$x^*(e_i) = \frac{r(S_i) - r(S_{i+1})^+}{a_{S_i,e_i}} \text{ for } i = 1, \ldots, k$$

Dual Greedy

$$\min_{x\geq 0}\{c^T x \mid \sum_{e\in S} a_{S,e}x_e \geq r(S)\ \forall S\subseteq E\} \qquad \max_{y\geq 0}\{r^T y \mid \sum_{S:\ e\in S} a_{S,e}y_S \leq c(e)\ \forall e\in E\}$$

Dual Greedy:

$y^* := \vec{0}, S := E$
while $r(S) > 0$ *and* $S \neq \varnothing$ **do**
  Raise $y^*(S)$ until some $e^* \in S$ gets tight
  $S := S \setminus \{e^*\}$
return y^*

Construct primal candidate: Based on chain $S_k \subset S_{k-1} \subset \ldots \subset S_1 = E$ and bottleneck elements $E^* = \{e_k, \ldots, e_1\}$ construct $x^* \in \mathbb{R}^E_+$ with non-zero values

$$x^*(e_i) = \frac{r(S_i) - r(S_{i+1})^+}{a_{S_i,e_i}} \text{ for } i = 1, \ldots, k$$

Note: y^* is feasible for the dual LP, and

$$x^*(e) > 0 \text{ implies } \sum_{S:e\in S} a_{S,e}y^*(S) = c(e).$$

Dual Greedy

$$\min_{x \geq 0}\{c^T x \mid \sum_{e \in S} a_{S,e} x_e \geq r(S)\ \forall S \subseteq E\} \qquad \max_{y \geq 0}\{r^T y \mid \sum_{S:\, e \in S} a_{S,e} y_S \leq c(e)\ \forall e \in E\}$$

Dual Greedy:

$y^* := \vec{0}, S := E$
while $r(S) > 0$ *and* $S \neq \emptyset$ **do**
  Raise $y^*(S)$ until some $e^* \in S$ gets tight
  $S := S \setminus \{e^*\}$
return y^*

Construct primal candidate: Based on chain $S_k \subset S_{k-1} \subset \ldots \subset S_1 = E$ and bottleneck elements $E^* = \{e_k, \ldots, e_1\}$ construct $x^* \in \mathbb{R}^E_+$ with non-zero values

$$x^*(e_i) = \frac{r(S_i) - r(S_{i+1})^+}{a_{S_i,e_i}} \text{ for } i = 1, \ldots, k$$

Note: y^* is feasible for the dual LP, and

$$x^*(e) > 0 \text{ implies } \sum_{S: e \in S} a_{S,e} y^*(S) = c(e).$$

Questions:

- Feasibility of x^*?
- Bounded performance guarantee of x^* and of $\hat{x} = (\lceil x^*(e) \rceil)_{e \in E}$?

Dual Greedy

$$\min_{x\geq 0}\{c^T x \mid \sum_{e\in S} a_{S,e} x_e \geq r(S)\ \forall S\subseteq E\} \qquad \max_{y\geq 0}\{r^T y \mid \sum_{S:\, e\in S} a_{S,e} y_S \leq c(e)\ \forall e\in E\}$$

Dual Greedy:

$y^* := \vec{0}, S := E$
while $r(S) > 0$ *and* $S \neq \emptyset$ **do**
  Raise $y^*(S)$ until some $e^* \in S$ gets tight
  $S := S \setminus \{e^*\}$
return y^*

Construct primal candidate: Based on chain $S_k \subset S_{k-1} \subset \ldots \subset S_1 = E$ and bottleneck elements $E^* = \{e_k, \ldots, e_1\}$ construct $x^* \in \mathbb{R}_+^E$ with non-zero values

$$x^*(e_i) = \frac{r(S_i) - r(S_{i+1})^+}{a_{S_i,e_i}} \text{ for } i = 1, \ldots, k$$

Note: y^* is feasible for the dual LP, and

$$x^*(e) > 0 \text{ implies } \sum_{S:e\in S} a_{S,e} y^*(S) = c(e).$$

Questions:

- Feasibility of x^*?
- Performance guarantee δ such that $\sum_{e\in S_i} a_{S_i,e} x^*(e) \leq \delta \cdot r(S_i)$ for all $S_i \in \text{supp}(y^*)$?

Approximate Complementary Slackness

Lemma

If x^* and y^* are feasible for the primal and dual, respectively, and if

(i) $x^*(e) > 0$ *implies* $\sum_{S:e\in S} a_{S,e} y^*(S) = c(e)$

(ii) $y^*(S) > 0$ *implies* $\sum_{e\in S} a_{S,e} x^*(e) \leq \delta \cdot r(S)$,

then x^* is a δ-approximation for the primal LP.

Approximate Complementary Slackness

Lemma

If x^* and y^* are feasible for the primal and dual, respectively, and if

(i) $x^*(e) > 0$ *implies* $\sum_{S:e\in S} a_{S,e} y^*(S) = c(e)$

(ii) $y^*(S) > 0$ *implies* $\sum_{e\in S} a_{S,e} x^*(e) \leq \delta \cdot r(S)$,

then x^* is a δ-approximation for the primal LP.

Reason:

$$
\begin{aligned}
c^T x^* &= \sum_{e\in E} c_e x^*(e) \overset{(i)}{=} \sum_{e\in E} \sum_{S\subseteq E} a_{S,e} y^*(S) x^*(e) \\
&= \sum_{S\subseteq E}^{k} y^*(S) \sum_{e\in S} a_{S,e} x^*(e) \overset{(ii)}{\leq} \delta \sum_{S\subseteq E} y^*(S) r(S) \leq \delta r^T y^* \leq \delta \cdot \mathrm{OPT}
\end{aligned}
$$

Feasibility is not guaranteed

$$\begin{aligned} \min x_1 + 2x_2 \\ 2x_1 + x_2 \geq 3 \\ x_2 \geq 1 \\ x_1 \geq 2 \\ 0 \geq 0 \end{aligned}$$

Feasibility is not guaranteed

$$\begin{aligned} \min x_1 + 2x_2 \\ 2x_1 + x_2 \geq 3 \\ x_2 \geq 1 \\ x_1 \geq 2 \\ 0 \geq 0 \end{aligned}$$

$$\begin{aligned} \max y_E + 2y_{\{1\}} + y_{\{2\}} \\ 2y_E + y_{\{1\}} \leq 1 \\ y_E + y_{\{2\}} \leq 2 \\ y \geq 0 \end{aligned}$$

Feasibility is not guaranteed

$$
\begin{aligned}
\min x_1 + 2x_2 \\
2x_1 + x_2 &\geq 3 \\
x_2 &\geq 1 \\
x_1 &\geq 2 \\
0 &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
\max y_E + 2y_{\{1\}} + y_{\{2\}} \\
2y_E + y_{\{1\}} &\leq 1 \\
y_E + y_{\{2\}} &\leq 2 \\
y &\geq 0
\end{aligned}
$$

Feasibility is not guaranteed

$$\begin{aligned} \min x_1 + 2x_2 \\ 2x_1 + x_2 &\geq 3 \\ x_2 &\geq 1 \\ x_1 &\geq 2 \\ 0 &\geq 0 \end{aligned}$$

$$\begin{aligned} \max y_E + 2y_{\{1\}} + y_{\{2\}} \\ 2y_E + y_{\{1\}} &\leq 1 \\ y_E + y_{\{2\}} &\leq 2 \\ y &\geq 0 \end{aligned}$$

Note: the primal-dual greedy returns

- dual solution y^* with non-zero entries $y^*_E = \frac{1}{2}$, and $y^*_{\{2\}} = \frac{3}{2}$, and
- primal candidate $x^* = (1, 1)$.

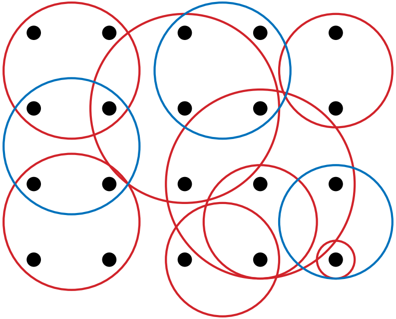
However, x^* is **infeasible** for $S = \{1\}$.

Primal Greedy Algorithm

Example: Set cover

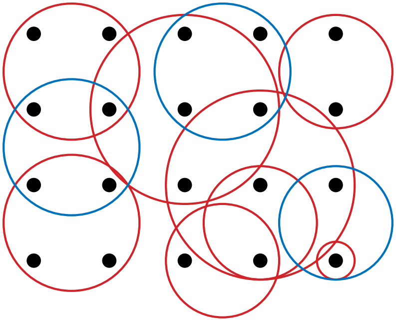
- Points V
- Subsets $T_i \subseteq V$ with costs c_i for $i \in [n] := E$
- Task: Find $S \subseteq E$ of minimum cost with $\bigcup_{i \in S} T_i = V$

$$\min \left\{ c^T x \mid \sum_{i: v \in T_i} x_i \geq 1 \ \forall v \in V, x \in \{0,1\}^E \right\}$$



Example: Set cover

- Points V
- Subsets $T_i \subseteq V$ with costs c_i for $i \in [n] := E$
- Task: Find $S \subseteq E$ of minimum cost with $\bigcup_{i \in S} T_i = V$



$$\min \left\{ c^T x \mid \sum_{i: v \in T_i} x_i \geq 1 \ \forall v \in V, x \in \{0,1\}^E \right\}$$

Equivalent formulation:

$$\min \left\{ c^T x \mid \sum_{e \in S} a_{S,e} x_e \geq r(S) \ \forall S \subseteq E, \ x \in \{0,1\}^E \right\},$$

where $r(S) = |V| - |\bigcup_{i \in E \setminus S} T_i|$, and $a_{S,e} = |T_e \setminus \bigcup_{i \in E \setminus S} T_i|$.

Primal Greedy (and Dual Fitting)

$$\min_{x \geq 0}\{c^T x \mid \sum_{e \in S} a_{S,e} x_e \geq r(S)\ \forall S \subseteq E\} \qquad \max_{y \geq 0}\{r^T y \mid \sum_{S:\, e \in S} a_{S,e} y_S \leq c(e)\ \forall e \in E\}$$

PrimalGreedy:

$\tilde{x} := \vec{0}, S := E$

while *Residual demand* $\tilde{r}(S) := r(S) - A_S \tilde{x} > 0$ **do**

  $\tilde{e} :=$ argmin$_{e \in S} \left\{ \frac{c(e)}{a_{S,e}} \mid a_{S,e} > 0 \right\}$

  $\tilde{x}(\tilde{e}) := \frac{r(S) - r(S \setminus \{\tilde{e}\})^+}{a_{S,e}}$

  $S := S \setminus \{\tilde{e}\}$

return $\tilde{x}$

Primal Greedy (and Dual Fitting)

$$\min_{x\geq 0}\{c^T x \mid \sum_{e\in S} a_{S,e}x_e \geq r(S)\ \forall S\subseteq E\} \qquad \max_{y\geq 0}\{r^T y \mid \sum_{S:\ e\in S} a_{S,e}y_S \leq c(e)\ \forall e\in E\}$$

PrimalGreedy:

$\tilde{x} := \vec{0}, S := E$
while *Residual demand* $\tilde{r}(S) := r(S) - A_S\tilde{x} > 0$ **do**
  $\tilde{e} :=$ argmin$_{e\in S}\left\{\frac{c(e)}{a_{S,e}} \mid a_{S,e} > 0\right\}$
  $\tilde{x}(\tilde{e}) := \frac{r(S)-r(S\setminus\{\tilde{e}\})^+}{a_{S,e}}$
  $S := S\setminus\{\tilde{e}\}$
return $\tilde{x}$

Note: Greedy constructs chain $S_k \subset \ldots \subset S_1 = E$ with $e_i = S_i \setminus S_{i+1}$ for $e_i \in \tilde{E} = \{e_k,\ldots,e_1\}$.

Primal Greedy (and Dual Fitting)

$$\min_{x \geq 0}\{c^T x \mid \sum_{e \in S} a_{S,e} x_e \geq r(S)\ \forall S \subseteq E\} \qquad \max_{y \geq 0}\{r^T y \mid \sum_{S:\ e \in S} a_{S,e} y_S \leq c(e)\ \forall e \in E\}$$

PrimalGreedy:

$\tilde{x} := \vec{0}, S := E$
while *Residual demand* $\tilde{r}(S) := r(S) - A_S \tilde{x} > 0$ **do**
  $\tilde{e} :=$argmin$_{e \in S} \left\{ \frac{c(e)}{a_{S,e}} \mid a_{S,e} > 0 \right\}$
  $\tilde{x}(\tilde{e}) := \frac{r(S) - r(S \setminus \{\tilde{e}\})^+}{a_{S,e}}$
  $S := S \setminus \{\tilde{e}\}$
return $\tilde{x}$

Note: Greedy constructs chain $S_k \subset \ldots \subset S_1 = E$ with $e_i = S_i \setminus S_{i+1}$ for $e_i \in \tilde{E} = \{e_k, \ldots, e_1\}$.

Construct dual candidate: $\tilde{y} \in \mathbb{R}_+^{2^E}$ with non-zero values

$$\tilde{y}(S_i) = \alpha_i - \alpha_{i-1} \text{ for } i = 1, \ldots, k,$$

where

$$\alpha_i := \begin{cases} \frac{c(e_i)}{a_{S_i,e_i}} & \text{if } i \in [k] \\ 0 & \text{if } i = 0. \end{cases}$$

Primal Greedy (and Dual Fitting)

$$\min_{x\geq 0}\{c^T x \mid \sum_{e\in S} a_{S,e}x_e \geq r(S)\ \forall S\subseteq E\} \qquad \max_{y\geq 0}\{r^T y \mid \sum_{S:\, e\in S} a_{S,e}y_S \leq c(e)\ \forall e\in E\}$$

PrimalGreedy:

$\tilde{x} := \vec{0}, S := E$
while *Residual demand* $\tilde{r}(S) := r(S) - A_S\tilde{x} > 0$ **do**
  $\tilde{e} :=$argmin$_{e\in S}\left\{\frac{c(e)}{a_{S,e}} \mid a_{S,e} > 0\right\}$
  $\tilde{x}(\tilde{e}) := \frac{r(S)-r(S\setminus\{\tilde{e}\})^+}{a_{S,e}}$
  $S := S \setminus \{\tilde{e}\}$
return $\tilde{x}$

Note: Greedy constructs chain $S_k \subset \ldots \subset S_1 = E$ with $e_i = S_i \setminus S_{i+1}$ for $e_i \in \tilde{E} = \{e_k, \ldots, e_1\}$.

Construct dual candidate: $\tilde{y} \in \mathbb{R}_+^{2^E}$ with non-zero values

$$\tilde{y}(S_i) = \alpha_i - \alpha_{i-1} \text{ for } i = 1, \ldots, k,$$

where

$$\alpha_i := \begin{cases} \frac{c(e_i)}{a_{S_i,e_i}} & \text{if } i\in[k] \\ 0 & \text{if } i = 0. \end{cases}$$

Note: $\tilde{y}$ might not be feasible for the dual LP.

Primal Greedy (and Dual Fitting)

$$\min_{x\geq 0}\{c^T x \mid \sum_{e\in S} a_{S,e}x_e \geq r(S)\ \forall S\subseteq E\} \qquad \max_{y\geq 0}\{r^T y \mid \sum_{S:\, e\in S} a_{S,e}y_S \leq c(e)\ \forall e\in E\}$$

PrimalGreedy:

$\tilde{x} := \vec{0}, S := E$
while *Residual demand* $\tilde{r}(S) := r(S) - A_S\tilde{x} > 0$ **do**
  $\tilde{e} :=$ argmin$_{e\in S}\left\{\frac{c(e)}{a_{S,e}} \mid a_{S,e} > 0\right\}$
  $\tilde{x}(\tilde{e}) := \frac{r(S)-r(S\setminus\{\tilde{e}\})^+}{a_{S,e}}$
  $S := S\setminus\{\tilde{e}\}$
return $\tilde{x}$

Note: Greedy constructs chain $S_k \subset \ldots \subset S_1 = E$ with $e_i = S_i \setminus S_{i+1}$ for $e_i \in \tilde{E} = \{e_k, \ldots, e_1\}$.

Construct dual candidate: $\tilde{y} \in \mathbb{R}_+^{2^E}$ with non-zero values

$$\tilde{y}(S_i) = \alpha_i - \alpha_{i-1} \text{ for } i = 1, \ldots, k,$$

where

$$\alpha_i := \begin{cases} \frac{c(e_i)}{a_{S_i,e_i}} & \text{if } i\in[k] \\ 0 & \text{if } i = 0. \end{cases}$$

Note: $\tilde{y}$ might not be feasible for the dual LP.

Lemma

The two vectors $\tilde{x}$ and $\tilde{y}$ satisfy $c^T\tilde{x} = r^T\tilde{y}$.

Lemma

The two vectors $\tilde{x}$ *and* $\tilde{y}$ *satisfy* $c^T\tilde{x} = r^T\tilde{y}$.

Lemma

The two vectors $\tilde{x}$ and $\tilde{y}$ satisfy $c^T\tilde{x} = r^T\tilde{y}$.

Reason:

$$c^T\tilde{x} = \sum_{e\in E} c_e\tilde{x}(e) = \sum_{i=1}^{k} c(e_i)\tilde{x}(e_i) = \sum_{i=1}^{k-1} c(e_i)\frac{r(S_i)-r(S_{i+1})}{a_{S_i,e_i}} + c(e_k)\frac{r(S_k)}{a_{S_k,e_k}}$$

$$= \underbrace{\frac{c(e_i)}{a_{S_1,e_1}}}_{\tilde{y}(S_1)} \cdot r(S_1) + \sum_{i=2}^{k} \underbrace{\left(\frac{c(e_i)}{a_{S_i,e_i}} - \frac{c(e_{i-1})}{a_{S_{i-1},e_{i-1}}}\right)}_{\tilde{y}(S_i)} \cdot r(S_i)$$

$$= \sum_{i=1}^{k} r(S_i)\tilde{y}(S_i) = r^T\tilde{y}.$$

Lemma

The two vectors $\tilde{x}$ and $\tilde{y}$ satisfy $c^T\tilde{x} = r^T\tilde{y}$.

Reason:

$$
\begin{aligned}
c^T\tilde{x} &= \sum_{e\in E} c_e \tilde{x}(e) = \sum_{i=1}^{k} c(e_i)\tilde{x}(e_i) = \sum_{i=1}^{k-1} c(e_i)\frac{r(S_i)-r(S_{i+1})}{a_{S_i,e_i}} + c(e_k)\frac{r(S_k)}{a_{S_k,e_k}} \\
&= \underbrace{\frac{c(e_i)}{a_{S_1,e_1}}}_{\tilde{y}(S_1)} \cdot r(S_1) + \sum_{i=2}^{k} \underbrace{\left(\frac{c(e_i)}{a_{S_i,e_i}} - \frac{c(e_{i-1})}{a_{S_{i-1},e_{i-1}}}\right)}_{\tilde{y}(S_i)} \cdot r(S_i) \\
&= \sum_{i=1}^{k} r(S_i)\tilde{y}(S_i) = r^T\tilde{y}.
\end{aligned}
$$

Questions:

- Feasibility of $\tilde{x}$?
- Scaling factor Δ such that $\frac{\tilde{y}}{\Delta}$ is feasible for the dual LP?

Lemma

The two vectors $\tilde{x}$ and $\tilde{y}$ satisfy $c^T\tilde{x} = r^T\tilde{y}$.

Reason:

$$c^T\tilde{x} = \sum_{e\in E} c_e\tilde{x}(e) = \sum_{i=1}^{k} c(e_i)\tilde{x}(e_i) = \sum_{i=1}^{k-1} c(e_i)\frac{r(S_i)-r(S_{i+1})}{a_{S_i,e_i}} + c(e_k)\frac{r(S_k)}{a_{S_k,e_k}}$$

$$= \underbrace{\frac{c(e_i)}{a_{S_1,e_1}}}_{\tilde{y}(S_1)} \cdot r(S_1) + \sum_{i=2}^{k} \underbrace{\left(\frac{c(e_i)}{a_{S_i,e_i}} - \frac{c(e_{i-1})}{a_{S_{i-1},e_{i-1}}}\right)}_{\tilde{y}(S_i)} \cdot r(S_i)$$

$$= \sum_{i=1}^{k} r(S_i)\tilde{y}(S_i) = r^T\tilde{y}.$$

Questions:

- Feasibility of $\tilde{x}$?
- Scaling factor Δ such that $\frac{\tilde{y}}{\Delta}$ is feasible for the dual LP? ($\longrightarrow$ Δ-approximation.)

Feasibility and Greedy Systems

Greedy Systems

Definition

Given finite set E, function $r\colon 2^E \to \mathbb{R}$, and weight matrix $A = [a_{S,e}]_{S,e} \in \mathbb{R}_+^{2^E \times E}$, we call r weighted supermodular w.r.t. A if for all $S \subseteq T \subseteq E$ and $e \in S$ with $a_{S,e}, a_{T,e} \neq 0$ holds

$$\frac{r(S) - r(S \setminus \{e\})}{a_{S,e}} \leq \frac{r(T) - r(T \setminus \{e\})}{a_{T,e}}.$$

Greedy Systems

Definition

Given finite set E, function $r\colon 2^E \to \mathbb{R}$, and weight matrix $A = [a_{S,e}]_{S,e} \in \mathbb{R}_+^{2^E \times E}$, we call r weighted supermodular w.r.t. A if for all $S \subseteq T \subseteq E$ and $e \in S$ with $a_{S,e}, a_{T,e} \neq 0$ holds

$$\frac{r(S) - r(S \setminus \{e\})}{a_{S,e}} \leq \frac{r(T) - r(T \setminus \{e\})}{a_{T,e}}.$$

Definition

System (A, r) is a greedy system if the following three properties hold:

(G1) $S \subseteq T \subseteq E$ implies $r(S) \leq r(T)$,

(G2) $S \subseteq T \subseteq E$ implies $a_{S,e} \leq a_{T,e}$ for all $e \in E$,

(G3) r is weighted supermodular w.r.t. A.

Example: Knapsack Cover

Knapsack Cover

Given: items E with values u_e, weights c_e, demand D.
Find: $S \subseteq E$ with $c(S) = \sum_{e \in S} c_e$ minimal and $u(S) = \sum_{e \in S} u_e \geq D$.

Example: Knapsack Cover

KNAPSACK COVER

Given: items E with values u_e, weights c_e, demand D.
Find: $S \subseteq E$ with $c(S) = \sum_{e \in S} c_e$ minimal and $u(S) = \sum_{e \in S} u_e \geq D$.

$$\begin{aligned} \text{minimize} \ & \sum_{e \in E} c_e x_e \\ \text{subject to} \ & \sum_{e \in E} u_e x_e \geq D \\ & x_e \in \{0,1\} \qquad e \in E \end{aligned}$$

Example: Knapsack Cover

KNAPSACK COVER

Given: items E with values u_e, weights c_e, demand D.
Find: $S \subseteq E$ with $c(S) = \sum_{e \in S} c_e$ minimal and $u(S) = \sum_{e \in S} u_e \geq D$.

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} c_e x_e \\ \text{subject to} \quad & \sum_{e \in E} u_e x_e \geq D \\ & x_e \in \{0,1\} \qquad e \in E \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} c_e x_e \\ \text{subject to} \quad & \sum_{e \in S} u_e x_e \geq D - \sum_{e \notin S} u_e \qquad S \subseteq E \\ & x_e \in \{0,1\} \qquad e \in E \end{aligned}$$

Example: Submodular Cover

Submodular Cover

Given: items E, weights c_e, submodular function $f : 2^E \to \mathbb{Z}_+$, demand D.
Find: $X \subseteq E$ with $c(X) = \sum_{e \in X} c_e$ minimal and $f(X) \geq D$.

Example: Submodular Cover

Submodular Cover

Given: items E, weights c_e, submodular function $f : 2^E \to \mathbb{Z}_+$, demand D.
Find: $X \subseteq E$ with $c(X) = \sum_{e \in X} c_e$ minimal and $f(X) \geq D$.

Lemma (Wolsey 82)

Submodular Cover *can be written as*

$$\min_{x \in \{0,1\}^E} \left\{ c^T x \mid \sum_{e \in S} a_{S,e} x_e \geq r(S) \ \forall S \subseteq E \right\},$$

where $a_{S,e} = f(E \setminus S \cup \{e\}) - f(E \setminus S)$ *and* $r(S) = D - f(E \setminus S)$.

Feasibility

Theorem

If (A, r) is a greedy system, then both greedy algorithms are guaranteed to construct feasible solutions x^* and $\tilde{x}$ of (P) for each $c \in \mathbb{R}_+^E$.

Feasibility

Theorem

If (A, r) is a greedy system, then both greedy algorithms are guaranteed to construct feasible solutions x^* and $\tilde{x}$ of (P) for each $c \in \mathbb{R}_+^E$.

- Note that all three properties that define a greedy system are required for the proof.

Feasibility

Theorem

If (A, r) is a greedy system, then both greedy algorithms are guaranteed to construct feasible solutions x^* and $\tilde{x}$ of (P) for each $c \in \mathbb{R}_+^E$.

- Note that all three properties that define a greedy system are required for the proof.
- The integrality gap can be arbitrarily bad for greedy systems.

Feasibility

Theorem

If (A, r) is a greedy system, then both greedy algorithms are guaranteed to construct feasible solutions x^* and $\tilde{x}$ of (P) for each $c \in \mathbb{R}_+^E$.

Feasibility

Theorem

If (A, r) is a greedy system, then both greedy algorithms are guaranteed to construct feasible solutions x^* and $\tilde{x}$ of (P) for each $c \in \mathbb{R}_+^E$.

Proof (sketch): We need to show for x^* (analog for $\tilde{x}$)

$$\sum_{e \in T} a_{T,e} x^*(e) \geq r(T) \quad \forall T \subseteq E \text{ with } r(T) > 0$$

Given chain $S_{k+1} \subseteq S_k \subseteq \ldots \subseteq S_1$, construct new chain $T_{k+1} \subseteq T_k \subseteq \ldots \subseteq T_2 \subseteq T_1$ by setting $T_1 = T$ and $T_{i+1} = T_i \setminus \{e_i\}$ for all $i \in [k]$.

Feasibility

Theorem

If (A, r) is a greedy system, then both greedy algorithms are guaranteed to construct feasible solutions x^* and $\tilde{x}$ of (P) for each $c \in \mathbb{R}_+^E$.

Proof (sketch): We need to show for x^* (analog for $\tilde{x}$)

$$\sum_{e \in T} a_{T,e} x^*(e) \geq r(T) \quad \forall T \subseteq E \text{ with } r(T) > 0$$

Given chain $S_{k+1} \subseteq S_k \subseteq \ldots \subseteq S_1$, construct new chain $T_{k+1} \subseteq T_k \subseteq \ldots \subseteq T_2 \subseteq T_1$ by setting $T_1 = T$ and $T_{i+1} = T_i \setminus \{e_i\}$ for all $i \in [k]$. Let $t \in [k]$ be the largest index such that $r(T_t) > 0$. Then

$$\sum_{e \in T} a_{T,e} x^*(e) = \sum_{i=1}^{k} a_{T,e_i} x^*(e_i) \geq \sum_{i=1}^{t} a_{T,e_i} x^*(e_i) = \sum_{i=1}^{t-1} a_{T,e_i} \frac{r(S_i) - r(S_{i+1})}{a_{S_i,e_i}} + a_{T,e_t} \cdot x^*(e_t)$$

$$\stackrel{(G3)}{\geq} \sum_{i=1}^{t-1} a_{T,e_i} \frac{r(T_i) - r(T_{i+1})}{a_{T_i,e_i}} + a_{T,e_t} \cdot x^*(e_t) \stackrel{(G2)}{\geq} r(T_1) - r(T_t) + a_{T,e_t} \cdot x^*(e_t).$$

Feasibility

Theorem

If (A, r) is a greedy system, then both greedy algorithms are guaranteed to construct feasible solutions x^* and $\tilde{x}$ of (P) for each $c \in \mathbb{R}_+^E$.

Proof (sketch): We need to show for x^* (analog for $\tilde{x}$)

$$\sum_{e \in T} a_{T,e} x^*(e) \geq r(T) \quad \forall T \subseteq E \text{ with } r(T) > 0$$

Given chain $S_{k+1} \subseteq S_k \subseteq \ldots \subseteq S_1$, construct new chain $T_{k+1} \subseteq T_k \subseteq \ldots \subseteq T_2 \subseteq T_1$ by setting $T_1 = T$ and $T_{i+1} = T_i \setminus \{e_i\}$ for all $i \in [k]$. Let $t \in [k]$ be the largest index such that $r(T_t) > 0$. Then

$$\sum_{e \in T} a_{T,e} x^*(e) = \sum_{i=1}^{k} a_{T,e_i} x^*(e_i) \geq \sum_{i=1}^{t} a_{T,e_i} x^*(e_i) = \sum_{i=1}^{t-1} a_{T,e_i} \frac{r(S_i) - r(S_{i+1})}{a_{S_i,e_i}} + a_{T,e_t} \cdot x^*(e_t)$$

$$\overset{(G3)}{\geq} \sum_{i=1}^{t-1} a_{T,e_i} \frac{r(T_i) - r(T_{i+1})}{a_{T_i,e_i}} + a_{T,e_t} \cdot x^*(e_t) \overset{(G2)}{\geq} r(T_1) - r(T_t) + a_{T,e_t} \cdot x^*(e_t).$$

Remains to show $a_{T,e_t} \cdot x^*(e_t) \geq r(T_t)$.

Proof (cont.)

Theorem

If (A, r) is a greedy system, then both greedy algorithms are guaranteed to construct feasible solutions x^* and $\tilde{x}$ of (P) for each $c \in \mathbb{R}_+^E$.

Proof (cont.): It remains to show that $a_{T,e_t} \cdot x^*(e_t) \geq r(T_t)$. We distinguish the two cases where $t = k$ and $t < k$.

- If $t = k$, then $r(S_t) - r(S_{t+1})^+ = r(S_t)$, and thus $x^*(e_t) = \frac{r(S_t)}{a_{S_k,e_t}}$. Moreover, (G3) together with the fact that $r(T_t \setminus \{e_t\}) \leq r(S_t \setminus \{e_t\}) \leq 0$ imply $\frac{r(S_t)}{a_{S_t,e_t}} \geq \frac{r(T_t)}{a_{T_t,e_t}}$. It follows that

$$a_{T,e_t} \cdot x^*(e_t) \overset{(G2)}{\geq} a_{T_t,e_t} \cdot \frac{r(S_t)}{a_{S_t,e_t}} \geq a_{T_t,e_t} \cdot \frac{r(T_t)}{a_{T_t,e_t}} = r(T_t).$$

- Else, if $t < k$, then $r(S_t) - r(S_{t+1})^+ = r(S_t) - r(S_{t+1})$, implying

$$a_{T,e_t} \cdot x^*(e_t) = a_{T,e_t} \cdot \frac{r(S_t) - r(S_{t+1})}{a_{S_t,e_t}} \underset{(G2)}{\overset{(G3)}{\geq}} a_{T_t,e_t} \cdot \frac{r(T_t) - r(T_{t+1})}{a_{T_t,e_t}} = r(T_t) - r(T_{t+1}) \geq r(T_t).$$

Integrality Gaps and Truncations

Integrality gap might be unbounded

Observation

Greedy system might have an unbounded integrality gap.

Integrality gap might be unbounded

Observation

Greedy system might have an unbounded integrality gap.

Consider KNAPSACK COVER with two items $\{1, 2\}$,large demand D, costs $c = (1, 0)$, and values $u = (D, D - 1)$.

Integrality gap might be unbounded

Observation

Greedy system might have an unbounded integrality gap.

Consider KNAPSACK COVER with two items $\{1,2\}$,large demand D, costs $c = (1, 0)$, and values $u = (D, D-1)$.

$$
\begin{array}{lllll}
\text{minimize} & x_1 & & & \\
\text{subject to} & Dx_1 & + & (D-1)x_2 & \geq & D \\
 & & & (D-1)x_2 & \geq & 0 \\
 & x_1, & & x_2 & \in & \{0,1\}
\end{array}
$$

Integrality gap might be unbounded

Observation

Greedy system might have an unbounded integrality gap.

Consider KNAPSACK COVER with two items $\{1, 2\}$,large demand D, costs $c = (1, 0)$, and values $u = (D, D - 1)$.

$$
\begin{array}{rlcrcr}
\text{minimize} & x_1 & & & & \\
\text{subject to} & Dx_1 & + & (D-1)x_2 & \geq & D \\
& & & (D-1)x_2 & \geq & 0 \\
& Dx_1 & & & \geq & 1 \\
& x_1, & & x_2 & \in & \{0,1\}
\end{array}
$$

Integrality gap might be unbounded

Observation

Greedy system might have an unbounded integrality gap.

Consider KNAPSACK COVER with two items $\{1, 2\}$,large demand D, costs $c = (1, 0)$, and values $u = (D, D - 1)$.

$$\begin{array}{lrcrcr} \text{minimize} & x_1 & & & & \\ \text{subject to} & Dx_1 & + & (D-1)x_2 & \geq & D \\ & & & (D-1)x_2 & \geq & 0 \\ & \color{red}{1}x_1 & & & \geq & 1 \\ & x_1, & & x_2 & \in & \{0,1\} \end{array}$$

Theorem (Carr et al. 00, Carnes & Shmoys '08)

Primal-dual greedy algorithm applied to

$$\min_{x \geq 0}\{c^T x \mid \sum_{e \in S} a_{S,e} x_e \geq r(S) \forall S \subseteq E\},$$

where $D - u(E \setminus S)$ *and* $a_{S,e} := \min\{u_e, r(S)^+\}$ *is a 2-approximation algorithm.*

Good idea: Truncations

Definition

Let (A, r) be a greedy system and let $A' = [a'_{S,e}]_{S,e}$ with

$$a'_{S,e} = \min\{a_{S,e}, r(S)^+ - r(S \setminus \{e\})^+\}$$

for all $S \subseteq E, e \in E$. Then we call (A', r) the truncation of greedy system (A, r).

Good idea: Truncations

Definition

Let (A, r) be a greedy system and let $A' = [a'_{S,e}]_{S,e}$ with

$$a'_{S,e} = \min\{a_{S,e}, r(S)^+ - r(S \setminus \{e\})^+\}$$

for all $S \subseteq E, e \in E$. Then we call (A', r) the truncation of greedy system (A, r).

Observation

- *Truncations of greedy systems are not necessarily greedy systems.*

Good idea: Truncations

Definition

Let (A, r) be a greedy system and let $A' = [a'_{S,e}]_{S,e}$ with

$$a'_{S,e} = \min\{a_{S,e}, r(S)^+ - r(S \setminus \{e\})^+\}$$

for all $S \subseteq E, e \in E$. Then we call (A', r) the truncation of greedy system (A, r).

Observation

- *Truncations of greedy systems are not necessarily greedy systems.*
- *The dual solution of either algorithm might be infeasible for the original system.*

Truncated systems still work

Lemma

Let (A', r) be the truncation of a greedy system (A, r). Then for all $x \in \mathbb{Z}_+^E$, it holds that

$$Ax \geq r \quad \textit{iff} \quad A'x \geq r.$$

Truncated systems still work

Lemma

Let (A', r) be the truncation of a greedy system (A, r). Then for all $x \in \mathbb{Z}_+^E$, it holds that

$$Ax \geq r \quad \textit{iff} \quad A'x \geq r.$$

Theorem

Let (A, r) be a greedy system with truncation (A', r). Then both algorithms applied to the truncated system and arbitrary cost $c\colon E \to \mathbb{R}_+$ return feasible solutions x^* and $\tilde{x}$ to (P).

Truncated systems still work

Lemma

Let (A', r) *be the truncation of a greedy system* (A, r). *Then for all* $x \in \mathbb{Z}_+^E$, *it holds that*

$$Ax \geq r \quad \textit{iff} \quad A'x \geq r.$$

Theorem

Let (A, r) *be a greedy system with truncation* (A', r). *Then both algorithms applied to the truncated system and arbitrary cost* $c\colon E \to \mathbb{R}_+$ *return feasible solutions* x^* *and* $\tilde{x}$ *to* (P).

Proof: Similar as above.

Performance Guarantees

Performance Guarantees

Given a truncation (A', r) of a greedy system (A, r), we can bound the approximation ratio of

- the dual greedy algorithm in terms of parameter δ, where

$$\delta = \max_{i \in [k]} \delta_i \quad \text{with} \quad \delta_i = \begin{cases} \frac{a'_{E,e}}{a'_{S_i,e_i}} \text{ if } i < k \\ 1 \text{ if } i = k \end{cases}$$

Performance Guarantees

Given a truncation (A', r) of a greedy system (A, r), we can bound the approximation ratio of

- the dual greedy algorithm in terms of parameter δ, where

$$\delta = \max_{i\in[k]} \delta_i \quad \text{with} \quad \delta_i = \begin{cases} \frac{a'_{E,e}}{a'_{S_i,e_i}} \text{ if } i < k \\ 1 \text{ if } i = k \end{cases}$$

- the primal greedy algorithm in terms of parameter $\Delta := \min\{\log(\hat{\delta}+1), H(\gamma)\}$, where

$$\begin{aligned} \hat{\delta} &= \max_{i\in[k]} \frac{a'_{E,e}}{a'_{S_i,e_i}} \\ \gamma &= \max_{e\in E} a'_{E,e}. \end{aligned}$$

Dual Greedy Bounds

Theorem

The dual greedy algorithm applied on the truncation (A', r) of a greedy system (A, r) with costs c yields fractional solution x^* and integral solution $\hat{x}$ with

$$c^T x^* \leq \delta \, \mathrm{OPT} \qquad \textit{and} \qquad c^T \hat{x} \leq 2\delta \, \mathrm{OPT} \, .$$

Dual Greedy Bounds

Theorem

The dual greedy algorithm applied on the truncation (A', r) of a greedy system (A, r) with costs c yields fractional solution x^* and integral solution $\hat{x}$ with

$$c^T x^* \leq \delta \, \mathrm{OPT} \qquad \textit{and} \qquad c^T \hat{x} \leq 2\delta \, \mathrm{OPT} \, .$$

Proof. Essentially approximate complementary slackness, weak duality, and the following lemma. □

Dual Greedy Bounds

Theorem

The dual greedy algorithm applied on the truncation (A', r) of a greedy system (A, r) with costs c yields fractional solution x^* and integral solution $\hat{x}$ with

$$c^T x^* \leq \delta\,\mathrm{OPT} \qquad \text{and} \qquad c^T \hat{x} \leq 2\delta\,\mathrm{OPT}\,.$$

Proof. Essentially approximate complementary slackness, weak duality, and the following lemma. □

Lemma

If the dual greedy is applied to (A', r) with arbitrary costs c, we obtain for every $S \subseteq E$ with $y^*(S) > 0$:

$$r(S) \leq \sum_{e \in S} a'_{S,e} x^*(e) \leq \delta r(S),$$

$$r(S) \leq \sum_{e \in S} a'_{S,e} \hat{x}(e) \quad \leq 2\delta r(S).$$

Dual Fitting Bound for Primal Greedy

- We cannot immediately use LP duality arguments since dual solution can be infeasible.

Dual Fitting Bound for Primal Greedy

- We cannot immediately use LP duality arguments since dual solution can be infeasible.
- Scale dual solution by $\Delta := \min\{1 + \log \hat{\delta}, H(\gamma)\}$, where $\gamma := \max_{e \in E} a'_{E,e}$ and $H(\gamma)$ denotes the γ-th harmonic number.

Dual Fitting Bound for Primal Greedy

- We cannot immediately use LP duality arguments since dual solution can be infeasible.
- Scale dual solution by $\Delta := \min\{1 + \log \hat{\delta}, H(\gamma)\}$, where $\gamma := \max_{e \in E} a'_{E,e}$ and $H(\gamma)$ denotes the γ-th harmonic number.

Lemma

$\frac{1}{\Delta} y^*$ *is feasible for* (D).

Dual Fitting Bound for Primal Greedy

- We cannot immediately use LP duality arguments since dual solution can be infeasible.
- Scale dual solution by $\Delta := \min\{1 + \log \hat{\delta}, H(\gamma)\}$, where $\gamma := \max_{e \in E} a'_{E,e}$ and $H(\gamma)$ denotes the γ-th harmonic number.

Lemma

$\frac{1}{\Delta} y^*$ *is feasible for* (D).

Dual Fitting Bound for Primal Greedy

- We cannot immediately use LP duality arguments since dual solution can be infeasible.
- Scale dual solution by $\Delta := \min\{1 + \log \hat{\delta}, H(\gamma)\}$, where $\gamma := \max_{e \in E} a'_{E,e}$ and $H(\gamma)$ denotes the γ-th harmonic number.

Lemma

$\frac{1}{\Delta} y^*$ *is feasible for* (D).

Proof: Similar as in Wolsey'82.

Dual Fitting Bound for Primal Greedy

- We cannot immediately use LP duality arguments since dual solution can be infeasible.
- Scale dual solution by $\Delta := \min\{1 + \log \hat{\delta}, H(\gamma)\}$, where $\gamma := \max_{e \in E} a'_{E,e}$ and $H(\gamma)$ denotes the γ-th harmonic number.

Lemma

$\frac{1}{\Delta} y^*$ *is feasible for* (D).

Proof: Similar as in Wolsey'82.

Theorem

The primal greedy applied to truncation (A', r) *of a greedy system* (A, r) *with costs* c *yields fractional solution* $\tilde{x}$ *and integral solution* $\hat{x}$ *with*

$$c^T \tilde{x} \leq \Delta \, \mathrm{OPT} \qquad \textit{and} \qquad c^T \hat{x} \leq 2\Delta \, \mathrm{OPT} \, .$$

Dual Fitting Bound for Primal Greedy

- We cannot immediately use LP duality arguments since dual solution can be infeasible.
- Scale dual solution by $\Delta := \min\{1 + \log \hat{\delta}, H(\gamma)\}$, where $\gamma := \max_{e \in E} a'_{E,e}$ and $H(\gamma)$ denotes the γ-th harmonic number.

Lemma

$\frac{1}{\Delta} y^*$ is feasible for (D).

Proof: Similar as in Wolsey'82.

Theorem

The primal greedy applied to truncation (A', r) of a greedy system (A, r) with costs c yields fractional solution $\tilde{x}$ and integral solution $\hat{x}$ with

$$c^T \tilde{x} \leq \Delta \, \mathrm{OPT} \qquad \textit{and} \qquad c^T \hat{x} \leq 2\Delta \, \mathrm{OPT}\,.$$

Proof: $\frac{1}{\Delta} c^T x^* = \frac{1}{\Delta} r^T y^* \leq \mathrm{OPT}.$ □

Limitations

- Unless NP $\subseteq$ DTIME$(n^{\mathcal{O}(\log \log n)})$, we cannot get better than a $\log n$-approximation.

Limitations

- Unless NP $\subseteq$ DTIME$(n^{\mathcal{O}(\log\log n)})$, we cannot get better than a $\log n$-approximation.
- Since we restrict dual solutions to have support only on a chain, we ignore a lot of solutions.

Limitations

- Unless NP $\subseteq$ DTIME$(n^{\mathcal{O}(\log \log n)})$, we cannot get better than a $\log n$-approximation.
- Since we restrict dual solutions to have support only on a chain, we ignore a lot of solutions.
- Integrality gap of truncations can still be linear in $|E|$.

Limitations

- Unless NP $\subseteq$ DTIME$(n^{\mathcal{O}(\log \log n)})$, we cannot get better than a $\log n$-approximation.
- Since we restrict dual solutions to have support only on a chain, we ignore a lot of solutions.
- Integrality gap of truncations can still be linear in $|E|$.

Thank You!