The pseudo-Boolean polytope and binary polynomial optimization

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Introduction

- Binary polynomial optimization is the problem of maximizing a multivariate polynomial function over the set of binary points (NP-hard in general).

- Based on the encoding of the polynomial function, we obtain two popular optimization problems: multilinear optimization and pseudo-Boolean optimization.

Multilinear optimization

- A hypergraph G is a pair (V, E), where V is a finite set of nodes and E is a set of edges, which are subsets of V of cardinality at least two.

- The rank of G is the maximum cardinality of any edge in E.

- With any $G = (V, E)$, and $c \in \mathbb{R}^{V \cup E}$, we associate the multilinear optimization problem:

$$\begin{aligned} \max \quad & \sum_{v \in V} c_v z_v + \sum_{e \in E} c_e \prod_{v \in e} z_v \\ \text{s.t.} \quad & z_v \in \{0,1\} \qquad \forall v \in V. \end{aligned} \tag{BPO$_\mathrm{m}$}$$

- Define $z_e := \prod_{v \in e} z_v$ for all $e \in E$:

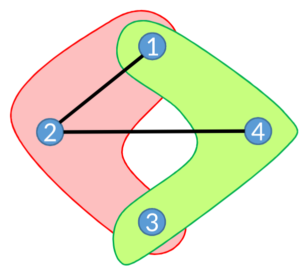
$$\begin{aligned} \max \quad & \sum_{v \in V} c_v z_v + \sum_{e \in E} c_e z_e, \\ \text{s.t.} \quad & z_e = \prod_{v \in e} z_v, \ \forall e \in E \\ & z_v \in \{0,1\}, \ \forall v \in V. \end{aligned} \tag{ℓBPO$_\mathrm{m}$}$$

Multilinear sets and polytopes

- We define the multilinear set as:

$$\mathcal{S}_{\mathrm{m}}(G) = \left\{ z \in \{0,1\}^{V \cup E} : z_e = \prod_{v \in e} z_v, \forall e \in E \right\}.$$

$$z_{12} = z_1 z_2$$
$$z_{24} = z_2 z_4$$
$$z_{123} = z_1 z_2 z_3$$
$$z_{134} = z_1 z_3 z_4$$



- The multilinear polytope $\mathcal{P}_{\mathrm{m}}(G)$ is the convex hull of the multilinear set.

- If $|e| = 2$ for all $e \in E$, then the objective function of Problem ($\mathrm{BPO_m}$) is quadratic and $\mathcal{P}_{\mathrm{m}}(G)$ is the Boolean quadric polytope $\mathrm{QP}(G)$ (Padberg, 89).

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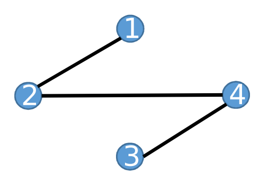
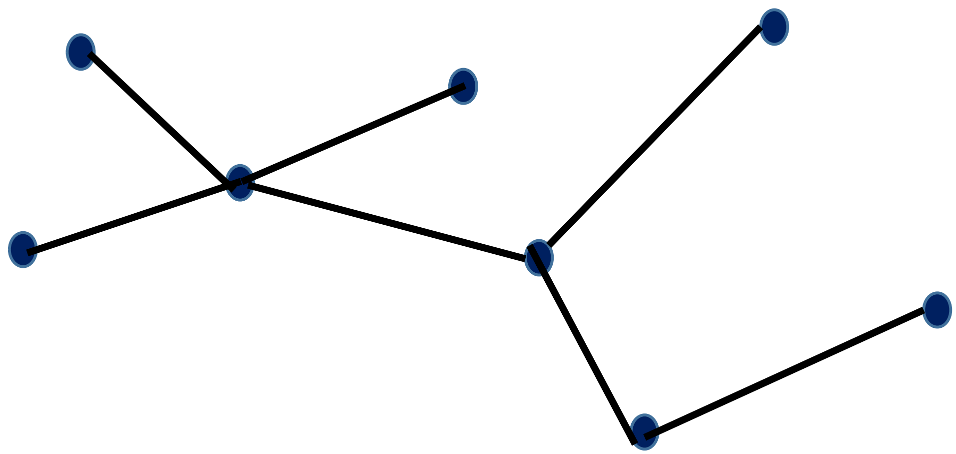


- Identify sufficient conditions under which $\mathcal{P}_{\mathrm{m}}(G)$ has a polynomial-size extended formulation.

Acyclic graphs

- Theorem (Padberg 89): The Boolean quadric polytope QP(G) of an acyclic graph $G = (V, E)$ has a formulation with $4|E|$ inequalities.



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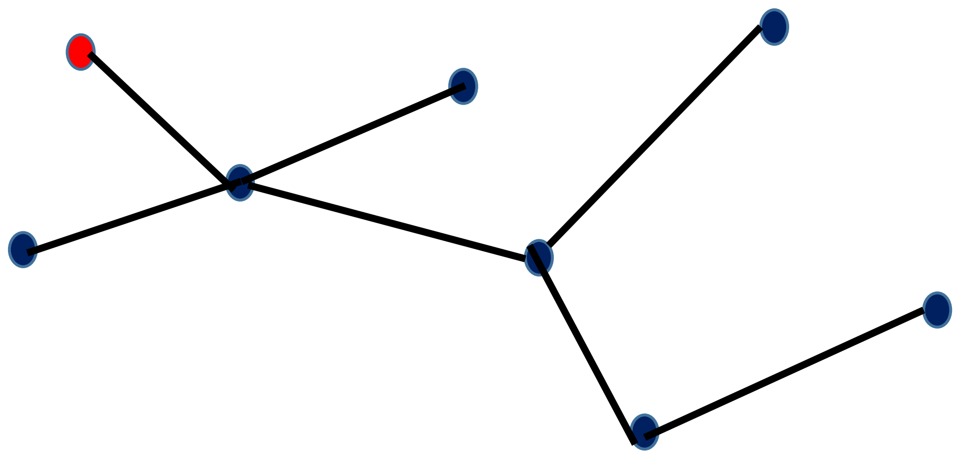

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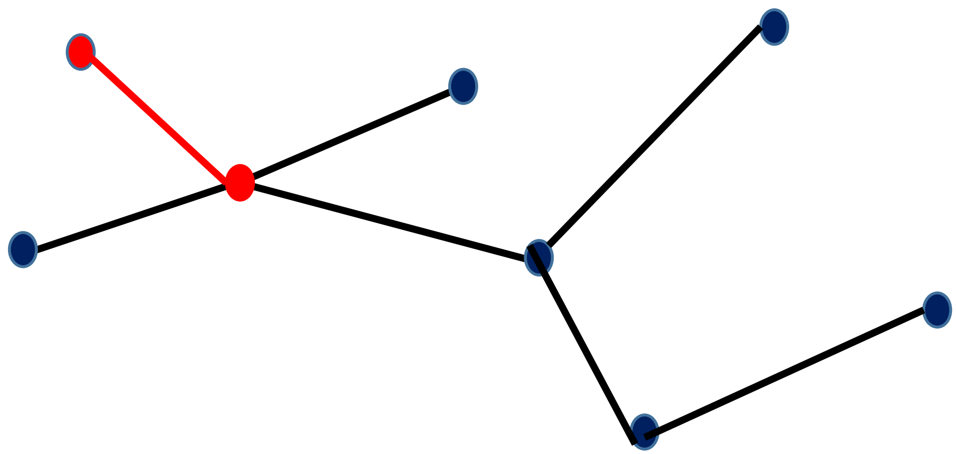

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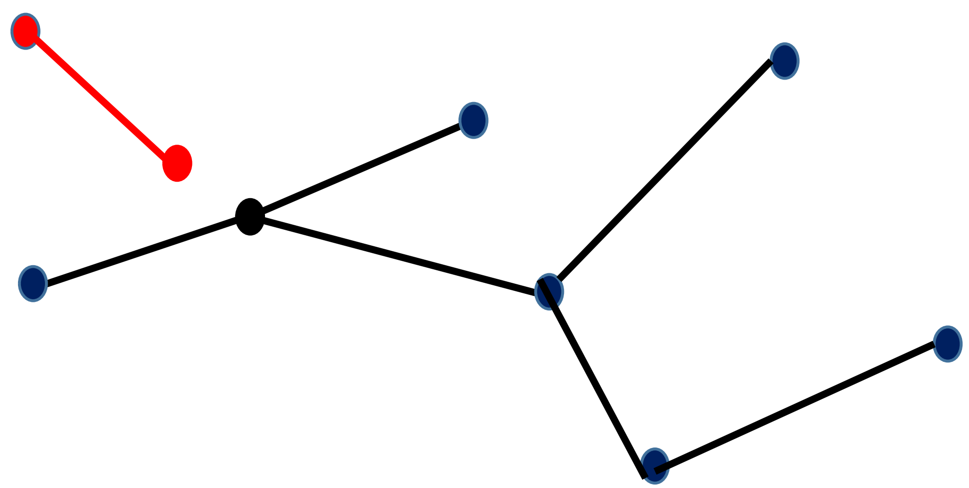

The multilinear polytope of acyclic hypergraphs

- Acyclic hypergraphs in increasing degree of generality:

$$\text{Berge} - \text{acyclic} \subset \gamma - \text{acyclic} \subset \beta - \text{acyclic} \subset \alpha - \text{acyclic}$$

- Optimizing over the multilinear polytope of α-acyclic hypergraphs is NP-hard in general.

- Theorem: The multilinear polytope of an α-acyclic hypergraph of rank r has an extended formulation with at most $O(2^r|V|)$ variables and inequalities.

- Equivalent to assuming bounded treewidth for the intersection graph: Wainwright-Jordan 2004, Laurent 2009, Bienstock-Munoz 2018.

α-acyclic hypergraphs

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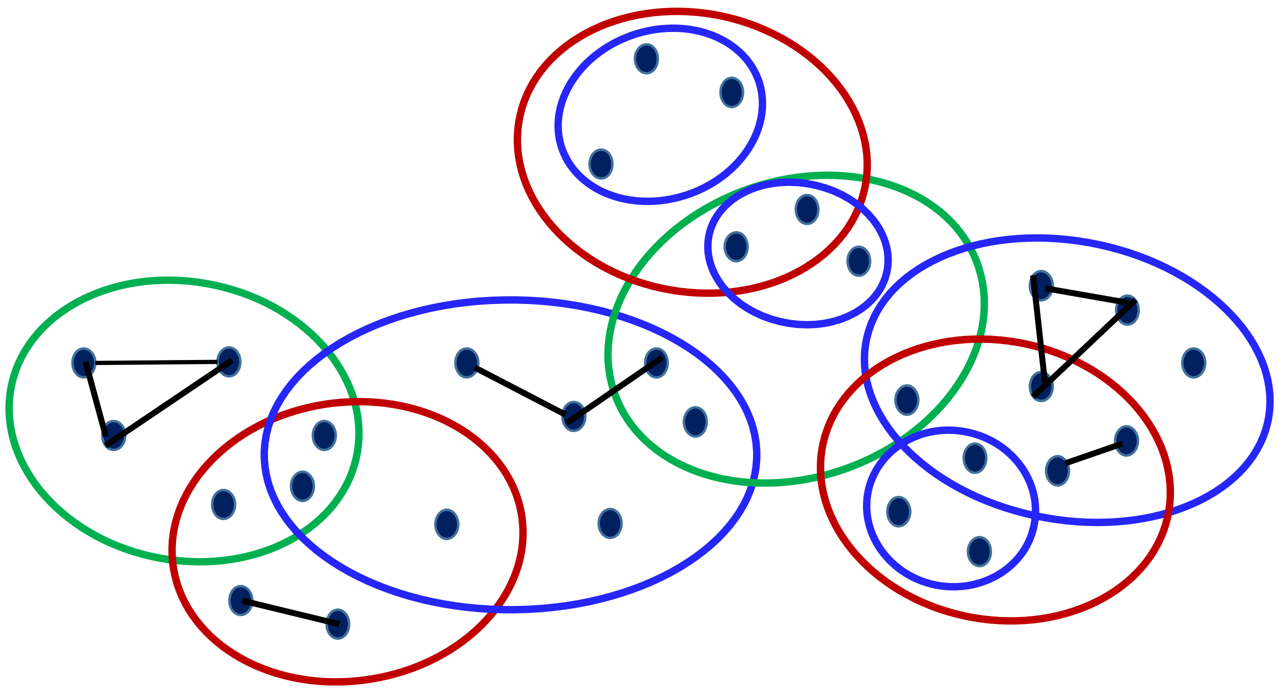

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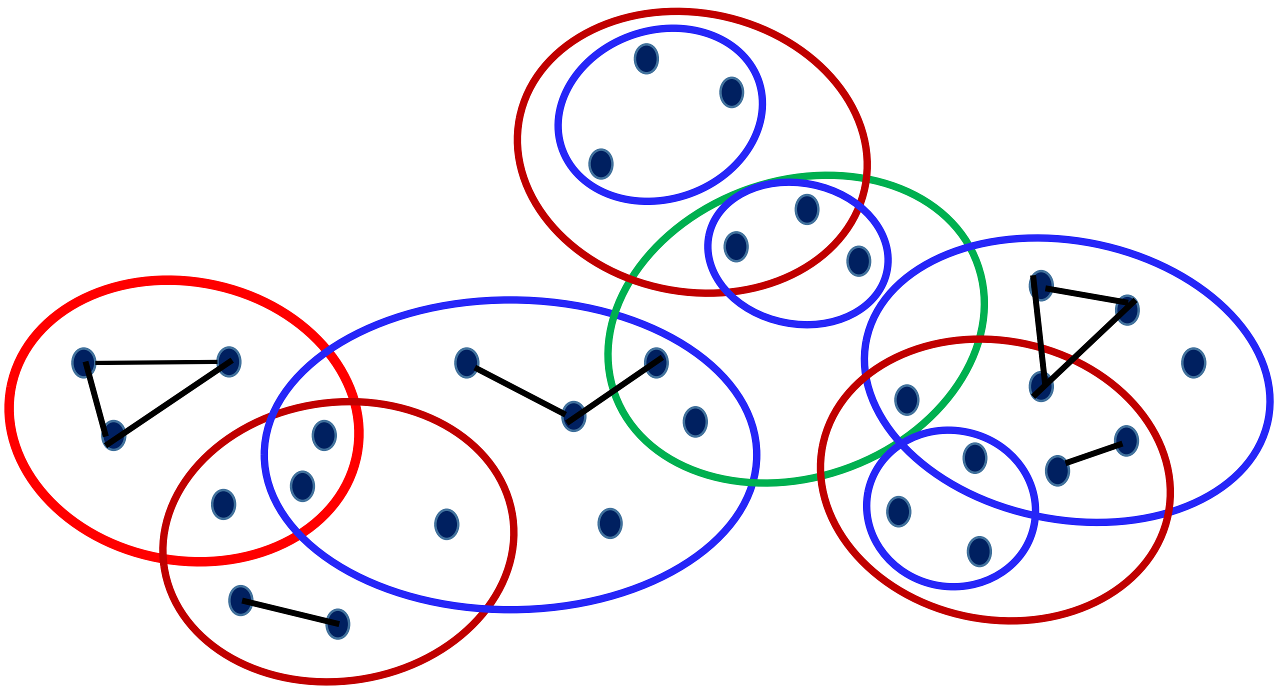

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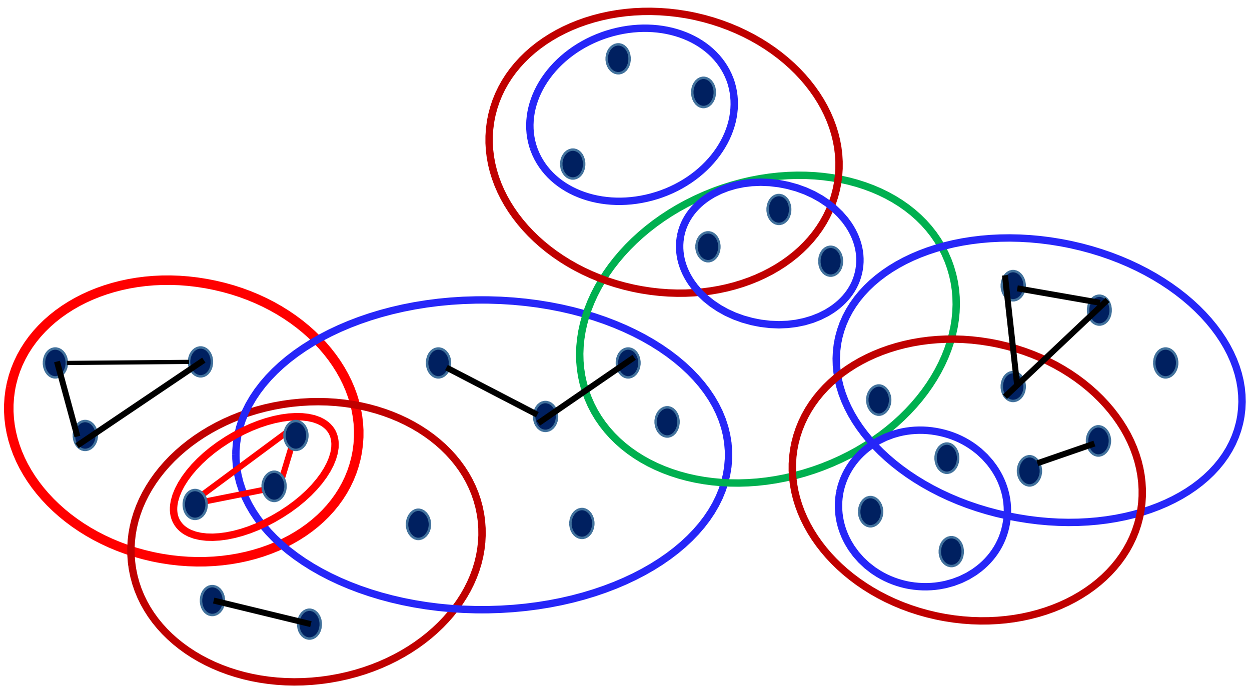

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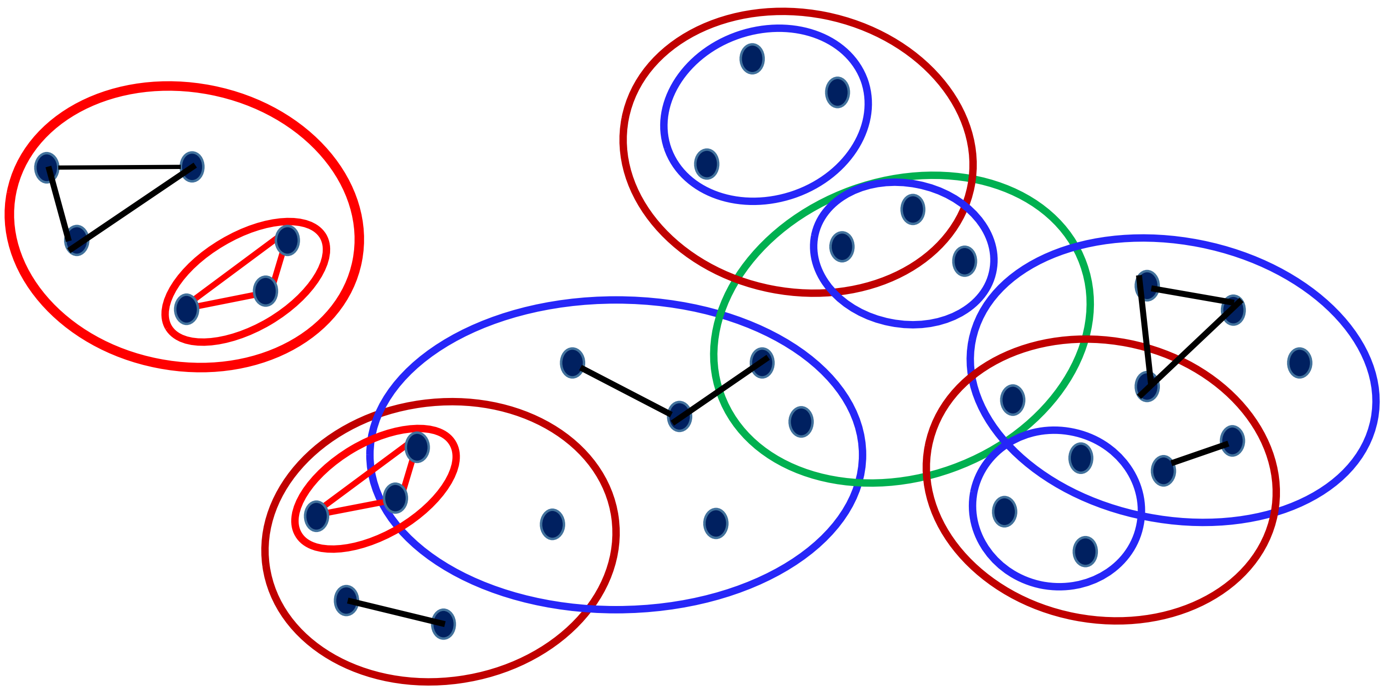
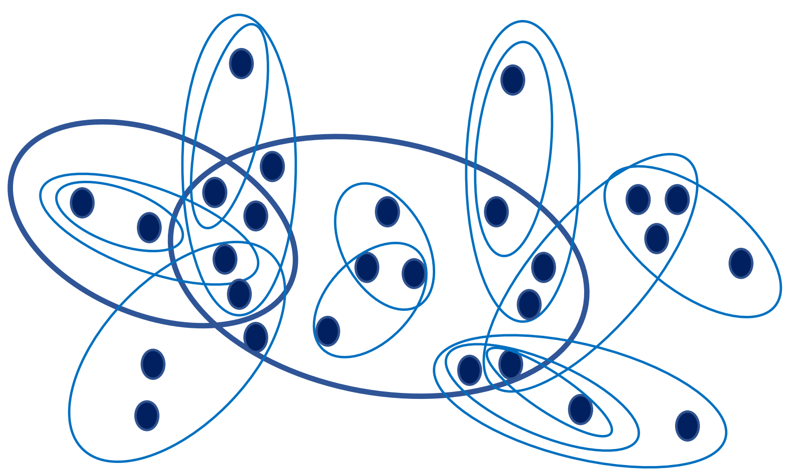


The multilinear polytope of β-acyclic hypergraphs

- A β-cycle of length q for $q \geq 3$ in G is a sequence $v_1, e_1, v_2, e_2, \ldots, v_q, e_q, v_1$ such that $v_1, v_2, \ldots, v_q$ are distinct nodes, $e_1, e_2, \ldots, e_q$ are distinct edges, and v_i belongs to e_{i-1}, e_i and no other edges for all $i = 1, \ldots, q$.

- **Theorem:** The multilinear polytope of β-acyclic hypergraphs has a polynomial-size extended formulation with at most $(r-1)|V| + |E|$ variables ($(r-2)|V|$ additional variables) and at most $(3r-4)|V| + 4|E|$ inequalities.

- The defining inequalities are very sparse: at most four variables with non-zero coefficients. All coefficients are ± 1 and all right-hand sides are $0/1$.



The multilinear polytope of β-acyclic hypergraphs

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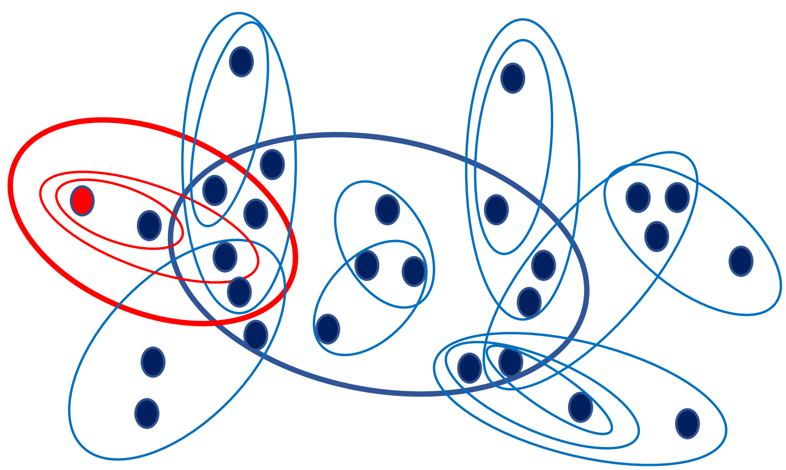

- A node is a β-leaf (nest point) if the edges containing it are totally ordered.

- A hypergraph is β-acyclic iff we can recursively remove β-leaves till obtaining an empty set.

The multilinear polytope of β-acyclic hypergraphs

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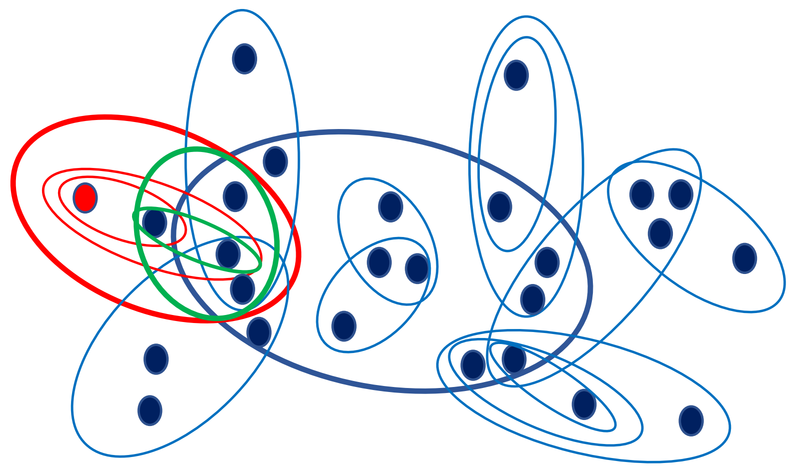

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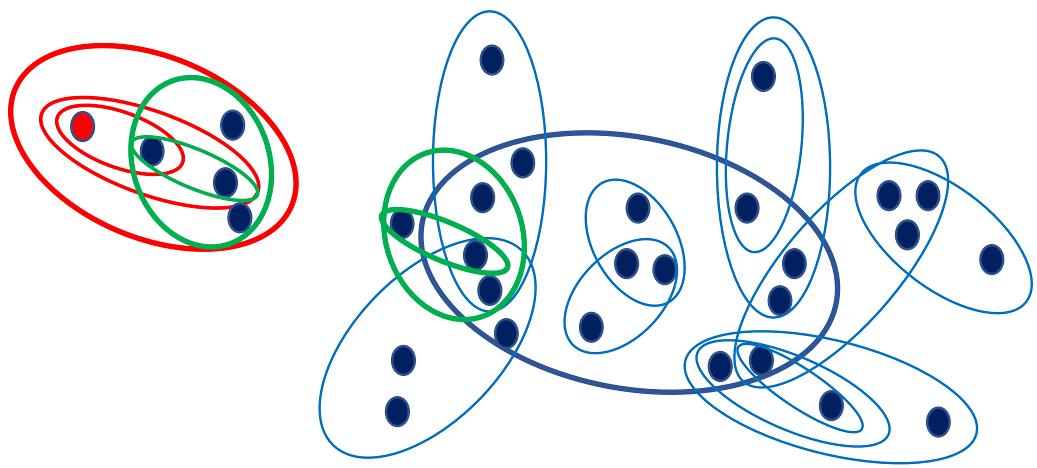


- The multilinear polytope of a pointed hypergraph $G = (V, E)$ consists of $5|V| + 2$ inequalities.

Beyond hypergraph acyclicity

- We present a new framework that
  - unifies all prior results on the existence of polynomial-size extended formulations, and
  - provides polynomial-size extended formulations for the multilinear polytope of hypergraphs with β-cycles

Signed hypergraphs

- A signed hypergraph H as a pair (V, S), where V is a finite set of nodes and S is a set of signed edges.

- A signed edge $s \in S$ is a pair (e, η_s), where e is a subset of V of cardinality at least two, and η_s is a map that assigns to each $v \in e$ a sign $\eta_s(v) \in \{-1, +1\}$.

- The underlying edge of a signed edge $s = (e, \eta_s)$ is e.

- Two signed edges $s = (e, \eta_s)$, $s' = (e', \eta_{s'}) \in S$ are parallel if $e = e'$, and they are identical if $e = e'$ and $\eta_s = \eta_{s'}$.

- We consider signed hypergraphs with no identical signed edges but often with parallel signed edges.

Pseudo-Boolean optimization

- With any signed hypergraph $H = (V, S)$, and cost vector $c \in \mathbb{R}^{V \cup S}$, we associate the pseudo-Boolean optimization problem:

$$\begin{aligned} \max \quad & \sum_{v \in V} c_v z_v + \sum_{s \in S} c_s \prod_{v \in s} \sigma_s(z_v) \\ \text{s.t.} \quad & z \in \{0,1\}^V, \end{aligned} \qquad (\mathrm{BPO_{pB}})$$

where

$$\sigma_s(z_v) := \begin{cases} z_v & \text{if } \eta_s(v) = +1 \\ 1 - z_v & \text{if } \eta_s(v) = -1. \end{cases}$$

- Define $z_s := \prod_{v \in s} \sigma_s(z_v)$ for all $s \in S$:

$$\begin{aligned} \max \quad & \sum_{v \in V} c_v z_v + \sum_{s \in S} c_s z_s, \\ \text{s.t.} \quad & z_s = \prod_{v \in s} \sigma_s(z_v), \ \forall s \in S \\ & z_v \in \{0,1\}, \ \forall v \in V. \end{aligned} \qquad (\ell\,\mathrm{BPO_{pB}})$$

Pseudo-Boolean sets and polytopes

- We define the pseudo-Boolean set of the signed hypergraph $H = (V, S)$, as:

$$\mathcal{S}_{\mathrm{pB}}(H) := \Big\{z \in \{0,1\}^{V \cup S} : z_s = \prod_{v \in s} \sigma_s(z_v), \ \forall s \in S\Big\},$$

  and we refer to its convex hull as the pseudo-Boolean polytope $\mathcal{P}_{\mathrm{pB}}(H)$.

- Unlike the multilinear polytope, the pseudo-Boolean polytope is NOT full dimensional.

The underlying hypergraph vs the multilinear hypergraph

- The underlying hypergraph of a signed hypergraph H is the hypergraph obtained from H by ignoring signs and dropping parallel edges.

- The pseudo-Boolean optimization problem over a signed hypergraph $H = (V, S)$ can be reformulated as a multilinear optimization problem over a hypergraph, which we call the multilinear hypergraph of H.

- Let the underlying hypergraph of H be β-acyclic; then the multilinear hypergraph of H may contain many β-cycles.

The recursive inflate and decompose framework

- Main ingredients:

1. A sufficient condition for decomposability of pseudo-Boolean polytopes.

2. A polynomial-size extended formulation for the pseudo-Boolean polytope of pointed signed hypergraphs, which appears as a result of applying the decomposition technique.

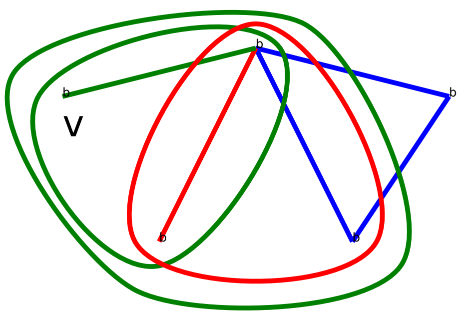
3. The inflation operation that we use to transform a large class of signed hypergraphs to those for which our results of Parts 1 and 2 are applicable.

Decomposability of pseudo-Boolean polytopes

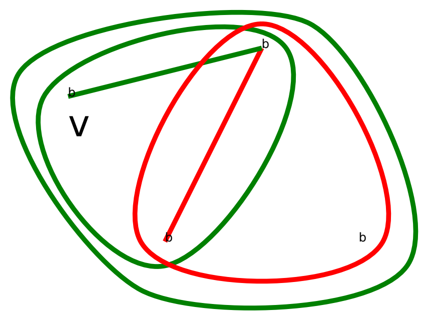
- Consider a signed hypergraph $H = (V, S)$, let $V_1, V_2 \subseteq V$ such that $V = V_1 \cup V_2$, let $S_1 \subseteq \{s \in S : s \subseteq V_1\}$, $S_2 \subseteq \{s \in S : s \subseteq V_2\}$ such that $S = S_1 \cup S_2$. Let $H_1 := (V_1, S_1)$ and $H_2 := (V_2, S_2)$.

- We say $\mathcal{P}_{\mathrm{pB}}(H)$ is decomposable into $\mathcal{P}_{\mathrm{pB}}(H_1)$ and $\mathcal{P}_{\mathrm{pB}}(H_2)$, if the system comprised of a description of $\mathcal{P}_{\mathrm{pB}}(H_1)$ and a description of $\mathcal{P}_{\mathrm{pB}}(H_2)$, is a description of $\mathcal{P}_{\mathrm{pB}}(H)$.

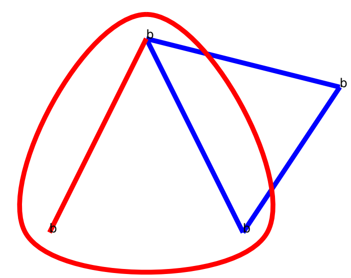
- Theorem: Assume the underlying hypergraph of H has a β-leaf v. Let $s_1 \subseteq s_2 \subseteq \cdots \subseteq s_k$ be the signed edges of H containing v, and assume S contains $s_i - v$ for all $i \in [k]$. Then $\mathcal{P}_{\mathrm{pB}}(H)$ is decomposable into $\mathcal{P}_{\mathrm{pB}}(H_1)$ and $\mathcal{P}_{\mathrm{pB}}(H_2)$, where $H_1 := (V_1, S_v \cup P_v)$, V_1 is the underlying edge of s_k, $S_v := \{s_1, \ldots, s_k\}$, $P_v := \{s_i - v : |s_i - v| \geq 2,\ i \in [k]\}$, and $H_2 := H - v$.



G



G_1



G_2

The pseudo-Boolean polytope of pointed signed hypergraphs

- Consider a signed hypergraph $H = (V, S)$ and let $v \in V$ be a β-leaf of the underlying hypergraph of H. Denote by S_v the set of all signed edges in S containing v. Define $P_v := \{s - v : s \in S_v, |s| \geq 3\}$. We say that H s a pointed signed hypergraph if V coincides with the underlying edge of the signed edge of maximum cardinality in S_v and $S = S_v \cup P_v$.

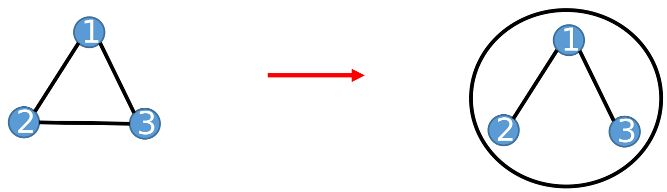
- Theorem: Let $H = (V, S)$ be a pointed signed hypergraph. Then $\mathcal{P}_{\text{pB}}(H)$ has a polynomial-size extended formulation with at most $2|V|(|S| + 1)$ variables and at most $4(|S|(|V| - 2) + |V|)$ inequalities. Moreover, all coefficients and right-hand side constants in the system defining $\mathcal{P}_{\text{pB}}(H)$ are $0, \pm 1$.

- Theorem: Let $H = (V, S)$ be a signed hypergraph of rank r whose underlying hypergraph is β-acyclic. Then the pseudo-Boolean polytope has a polynomial-size extended formulation with at most $O(r|S||V|)$ variables and inequalities.

Inflation of signed edges

- Let $H = (V, S)$ be a signed hypergraph, let $s \in S$, and let $e \subseteq V$ such that $s \subset e$. let $I(s, e)$ be the set of all possible signed edges s' parallel to e such that $\eta_s(v) = \eta_{s'}(v)$ for every $v \in s$. Then $H' = (V, S')$ is obtained from H by inflating s to e if $S' = S \cup I(s, e) \setminus \{s\}$. We say H' is obtained from H via an inflation operation.

- Theorem: Let $H' = (V, S')$ be obtained from H by inflating s to e. Then an extended formulation of $\mathcal{P}_{\mathrm{pB}}(H)$ can be obtained by juxtaposing an extended formulation of $\mathcal{P}_{\mathrm{pB}}(H')$ and $z_s = \sum_{s' \in I(s,e)} z_{s'}$. If $\mathcal{P}_{\mathrm{pB}}(H')$ has a polynomial-size extended formulation and $|e| - |s| = O(\log \mathrm{poly}(|V|, |S|))$, then $\mathcal{P}_{\mathrm{pB}}(H)$ has a polynomial-size extended formulation as well.



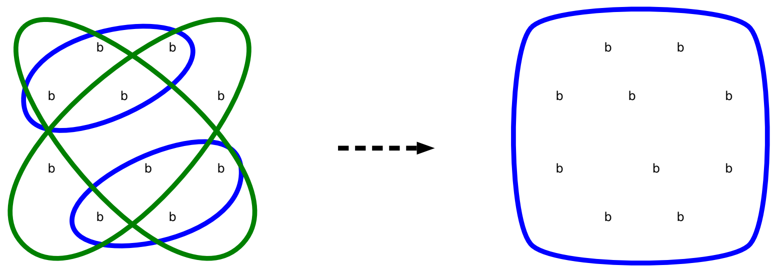
$s_1 = \{v_1^+, v_2^+\}$, $s_2 = \{v_1^+, v_3^+\}$, $s_3 = \{v_3^+, v_2^+\}$

$s_4 = \{v_1^-, v_2^+, v_3^+\}$, $s_5 = \{v_1^+, v_2^+, v_3^+\}$

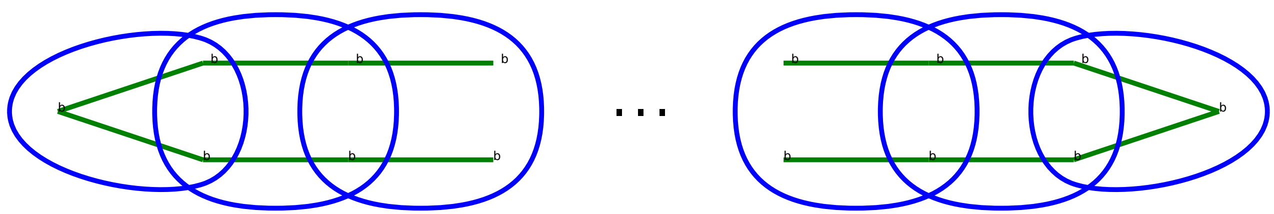
$z_{s_3} = z_{s_4} + z_{s_5}$

Applications of inflation

- Consider a signed hypergraph $H = (V, S)$. Suppose that each $s \in S$ contains at least $|V| - k$ nodes. Then the pseudo-Boolean polytope has an extended formulation with $O(2^k |V||S|)$ variables and inequalities.



- Consider a signed hypergraph $H = (V, S)$ of rank r. For each $s \in S$, among all maximal signed edges of H containing s, denote by f_s one with minimum cardinality. Let k be such that $|f_s| - |s| \leq k$ for all $s \in S$. Let $\bar{S}$ denote the set of maximal signed edges of H. If the underlying hypergraph of $(V, \bar{S})$ is β-acyclic, then the pseudo-Boolean polytope has an extended formulation with $O(r 2^k |V||S|)$ variables and inequalities.



The Recursive inflate-and-decompose (RID) framework

- Input: A signed hypergraph $H = (V, S)$, Output: An extended formulation for $\mathcal{P}_{\rm pB}(H)$.

- Step 0. Set $H^{(0)} := H$, $i := 0$.

- Step 1. If we can obtain $\bar{H}^{(i)}$ from $H^{(i)}$ via a number of inflation operations, such that a suitable extended formulation for $\mathcal{P}_{\rm pB}(\bar{H}^{(i)})$ is available, then we are done. Otherwise, go to Step 2.

- Step 2. Choose a node $\bar{v}$ of $H^{(i)}$. If $\bar{v}$ is a β-leaf of the underlying hypergraph of $H^{(i)}$, then set $\bar{H}^{(i)} := H^{(i)}$ and go to Step 3. Otherwise, construct $\bar{H}^{(i)}$ from $H^{(i)}$ via inflation operations, such that v is a β-leaf of the underlying hypergraph of $\bar{H}^{(i)}$. It suffices to find an extended formulation for $\mathcal{P}_{\rm pB}(\bar{H}^{(i)})$.

- Step 3. Decompose $\mathcal{P}_{\rm pB}(\bar{H}^{(i)})$ into $\mathcal{P}_{\rm pB}(\bar{H}_1^{(i)})$ and $\mathcal{P}_{\rm pB}(\bar{H}_2^{(i)})$, where $\bar{H}_1^{(i)}$ denotes the signed hypergraph containing node $\bar{v}$. Since we have an extended formulation for $\mathcal{P}_{\rm pB}(\bar{H}_1^{(i)})$, it suffices to find an extended formulation for $\mathcal{P}_{\rm pB}(\bar{H}_2^{(i)})$. Set $H^{(i+1)} := \bar{H}_2^{(i)}$, increment i by one, and go to Step 1.

α-acyclic hypergraphs with log-poly ranks

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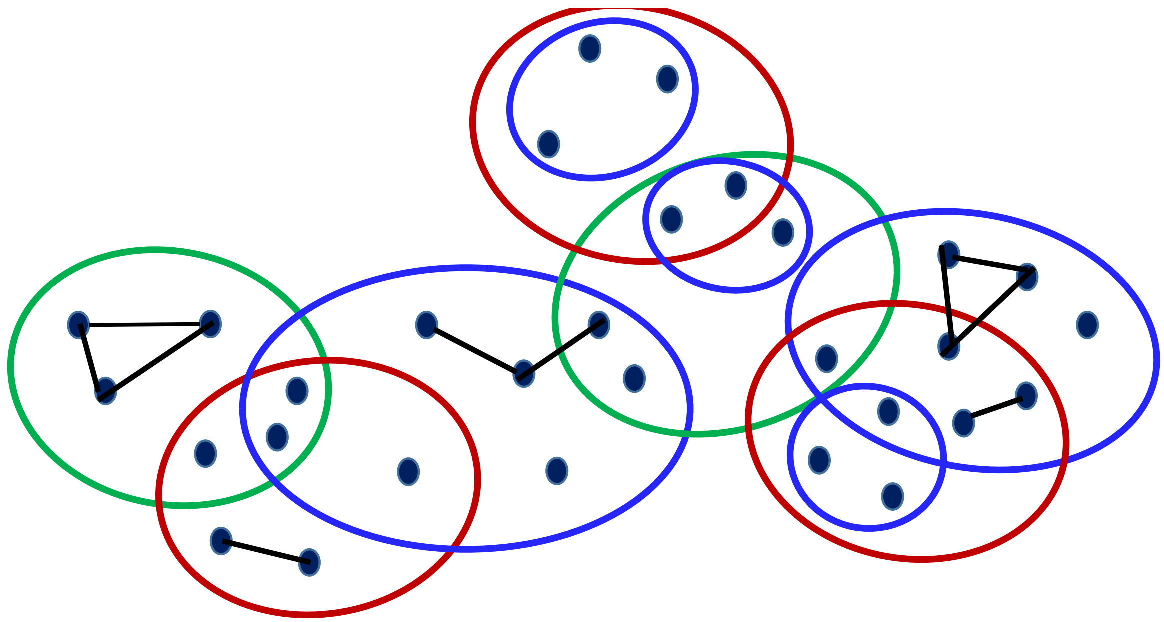

- A node $v \in V$ is an α-leaf if the set of edges containing v has a maximal element for inclusion.

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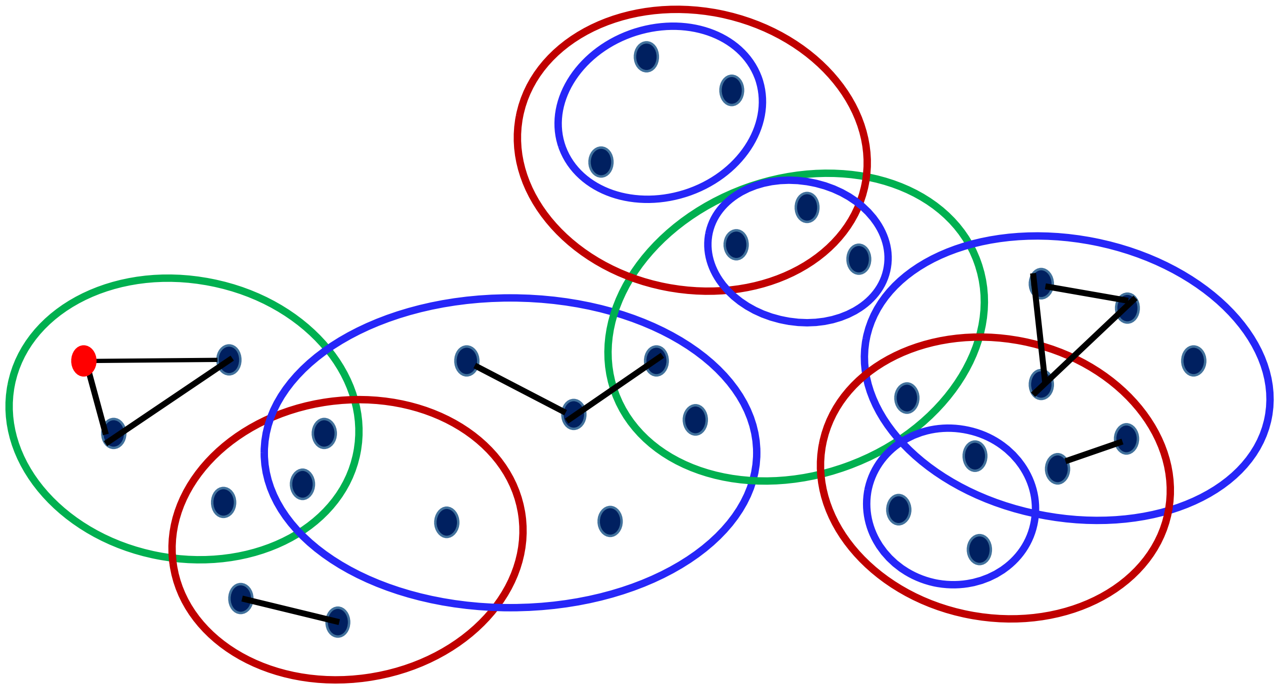

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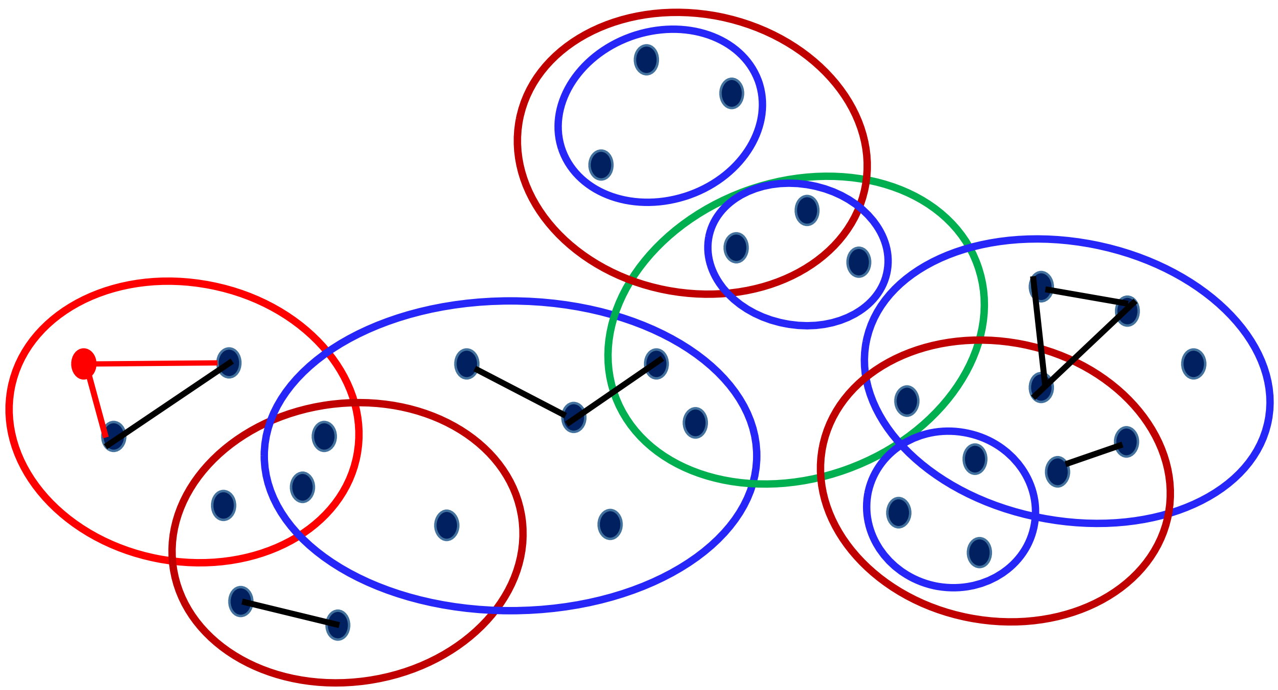

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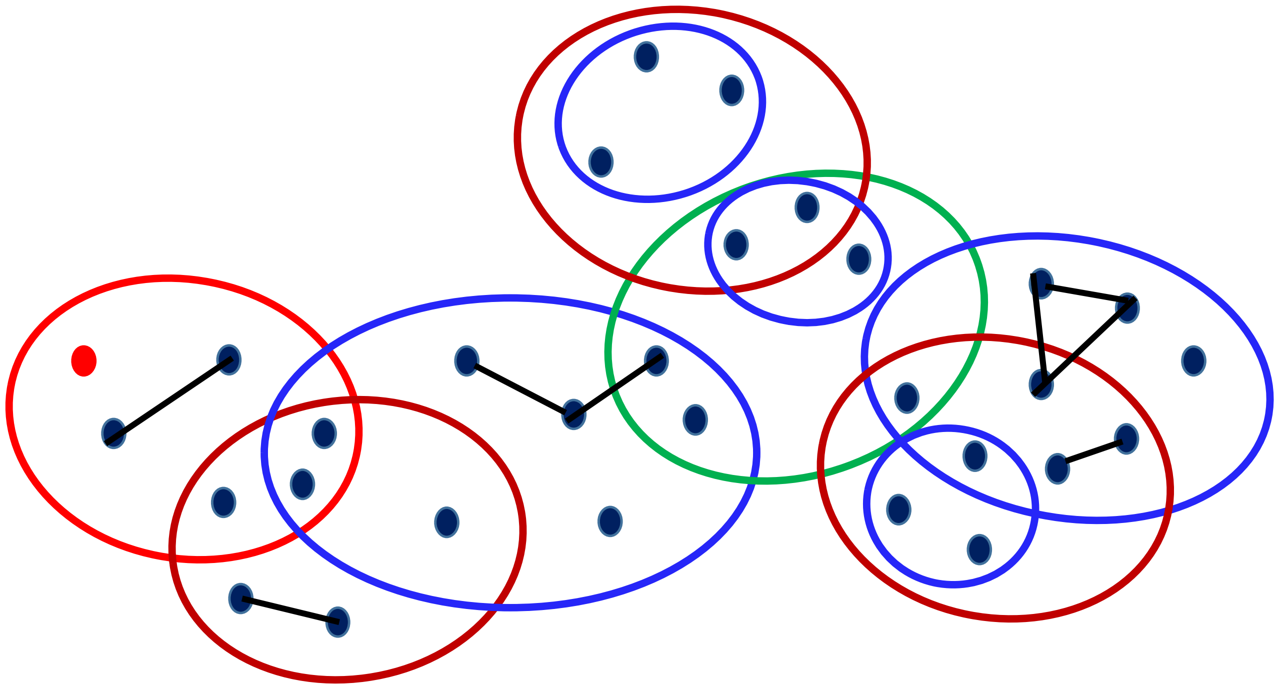

Nest-sets

- Let $G = (V, E)$ be a hypergraph and let $N \subseteq V$. Let $F(N)$ the set of edges in E containing some $v \in N$; N is a nest-set of G, if the set

$$F \setminus N := \{e \setminus N : e \in F(N)\},$$

  is totally ordered with respect to inclusion. If $|N| = 1$, then N contains a nest point of G.

- Let $N_i \subseteq V$, for all $i \in [t]$ and for some $t \geq 1$ such that the sets N_i are pairwise disjoint and $\cup_{i \in [t]} N_i = V$. We say that $\mathcal{N} = N_1, \cdots, N_t$ is a sequence of nest-sets of G, if N_1 is a nest set of G, N_2 is a nest-set of $G - N_1$, and so on.

- Given a sequence of nest-sets $\mathcal{N}$ of G, the nest-set width of this sequence $\mathrm{nsw}_{\mathcal{N}}(G)$, is the maximum cardinality of any element in $\mathcal{N}$.

- The nest-set width of G $\mathrm{nsw}(G)$, is the minimum value of $\mathrm{nsw}_{\mathcal{N}}(G)$ over all nest-set sequences $\mathcal{N}$ of G.

- $\mathrm{nsw}(G) = 1$, if and only if G is a β-acyclic hypergraph.

Hypergraphs with small nest-set widths

- Deciding if $\mathrm{nsw}(G) \le k$ for any integer k is NP-complete. However, when parameterized by k, this problem is fixed-parameter tractable (Lanzinger 2023):

- There exists a $2^{O(k^2)}\mathrm{poly}(|V|, |E|)$ time algorithm that takes as input hypergraph $G = (V, E)$ and integer $k \ge 1$ and returns a nest-set sequence $\mathcal{N}$ with $\mathrm{nsw}_{\mathcal{N}}(G) = k$ if one exists, or rejects otherwise.

- Theorem: Let $H = (V, S)$ be a signed hypergraph whose underlying hypergraph $G = (V, E)$ satisfies $\mathrm{nsw}(G) \le k$. Then the pseudo-Boolean polytope $\mathcal{P}_{\mathrm{pB}}(H)$ has an extended formulation with $O(2^k|V|^2|S|)$ variables and inequalities.

Hypergraphs with small nest-set widths

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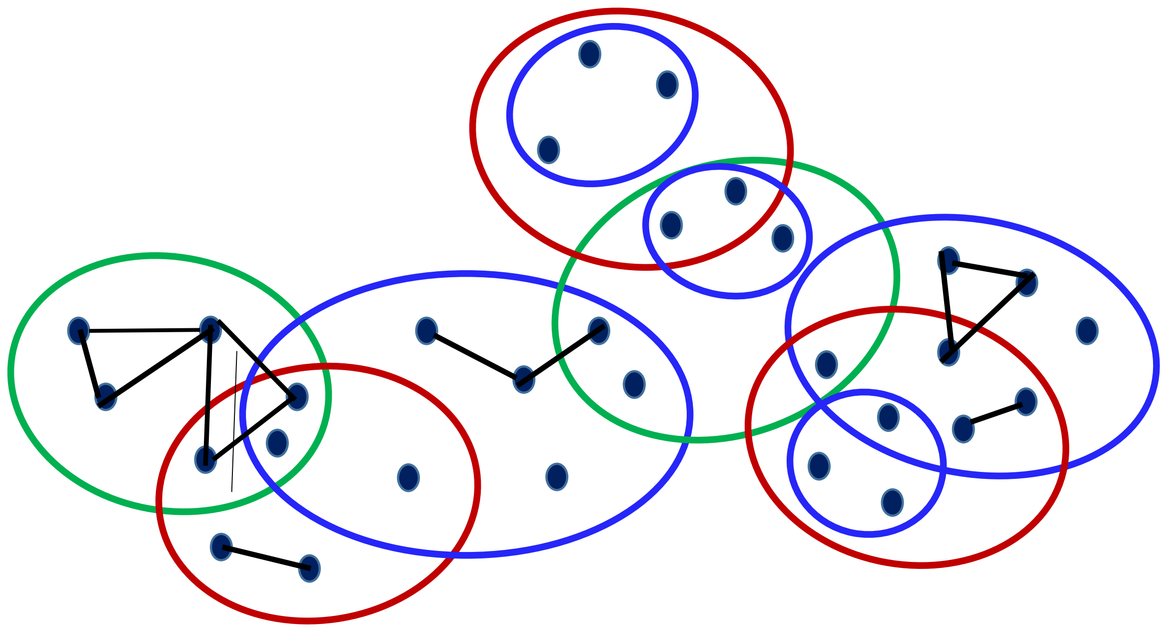

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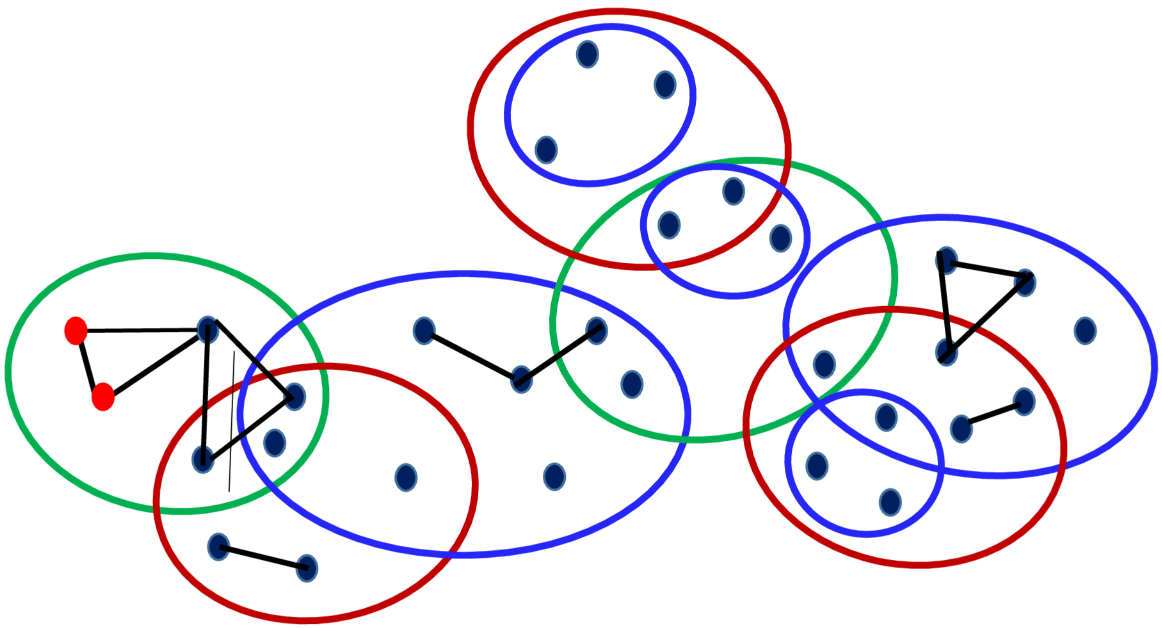

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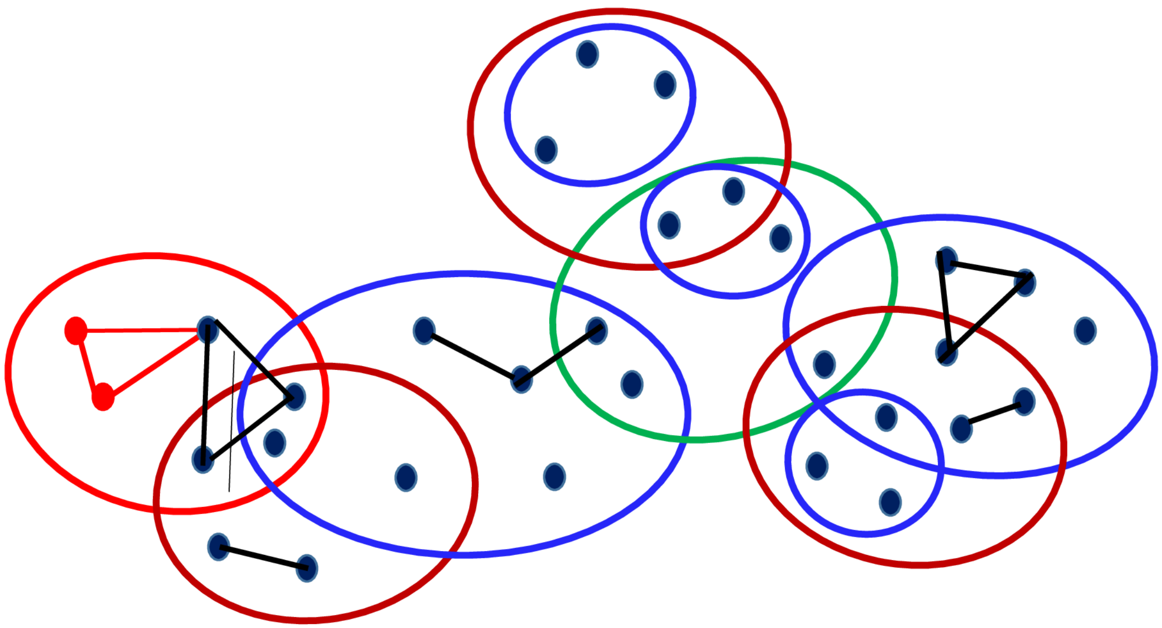

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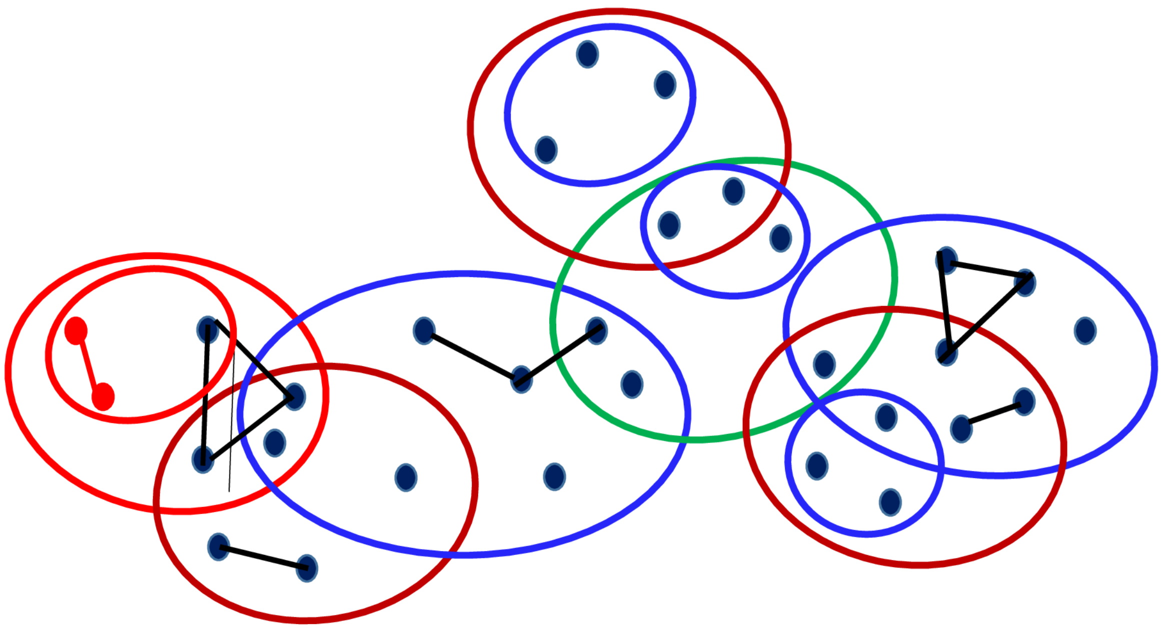

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