On Breaking k-Trusses

Dutch Seminar on Optimization

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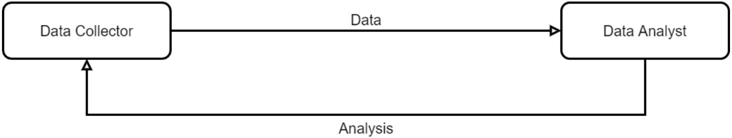
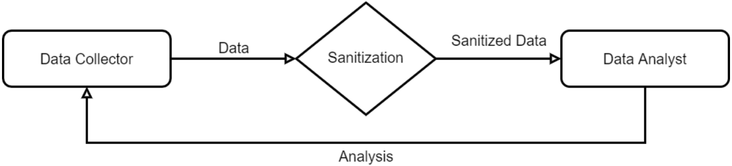
25th March 2021

Data Sanitization

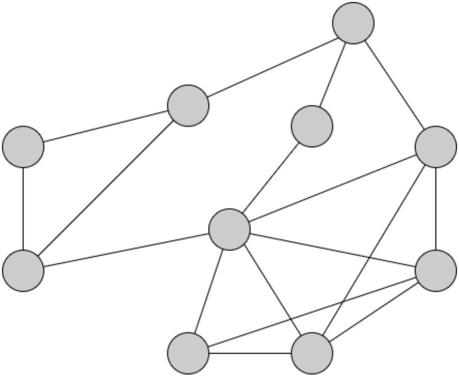


Data Sanitization

- Privacy Constraints
- Utility Properties



Communities



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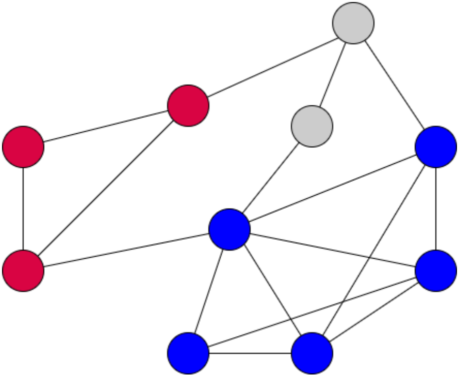

Community Breaking

- Maintaining communities in social networks

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- Assessing resilience to attacks and errors in communication networks

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- Assessing resilience to attacks and errors in communication networks
- Hiding membership to communities in social networks
- Preventing detection of confidential communities

k-Trusses

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A k-truss is a subgraph in which each edge is contained in at least $k-2$ triangles of the subgraph.

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Find a minimum set of edges to delete to remove all k-trusses.

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Problem (MIN-k-CBS)

Find a minimum set of edges incident to U such that no node in U is in a k-truss.

NP-hardness

Theorem

The problems MIN-k-TBS and MIN-k-CBS are NP-hard.

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For $\epsilon > 0$, both MIN-k-TBS and MIN-k-CBS cannot be approximated within a multiplicative factor of $(k - 2 - \epsilon)$, assuming the unique games conjecture.

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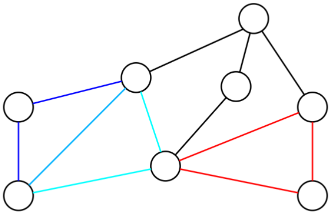
- Let G be an instance of MIN-3-TBS.



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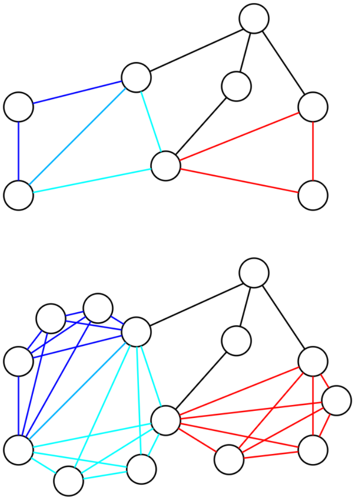
- Turn each triangle in G into a k-clique by adding $k - 3$ vertices to obtain G'.



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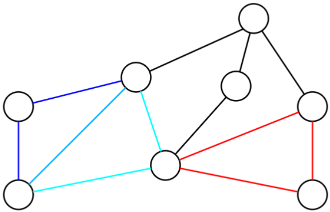

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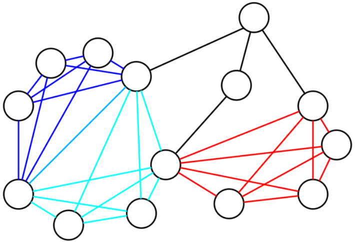
- Let G be an instance of MIN-3-TBS.
- Turn each triangle in G into a k-clique by adding $k - 3$ vertices to obtain G'.
- MIN-k-TBS in G' is equivalent to MIN-3-TBS in G.



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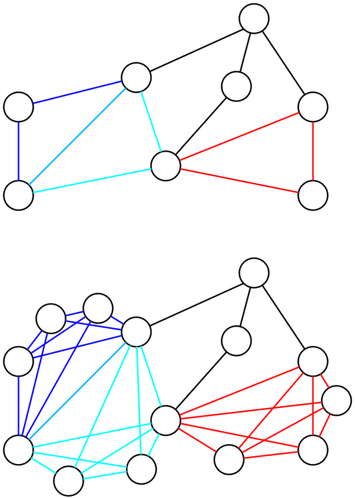
- Let G be an instance of MIN-3-TBS.
- Turn each triangle in G into a k-clique by adding $k - 3$ vertices to obtain G'.
- MIN-k-TBS in G' is equivalent to MIN-3-TBS in G.

Therefore MIN-k-TBS is NP-hard. $\square$



Algorithms

- Exact Algorithm

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- Heuristics

Truss Decomposition

Definition (k-Truss)

A k-truss is a subgraph in which each edge is contained in at least $k-2$ triangles of the subgraph.

Algorithm

$k = 2$ (every graph G is a 2-truss)
while G non-empty:

- $k = k + 1$
- while there is an edge in less than $k-2$ triangles:
  - remove that edge
- G is currently the maximum k-truss

Dynamic Truss Update for Edge Deletion

Definition (Triangle trussness)

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Update Algorithm

- Similar Algorithm
- Store the triangles each edge is in.
- Store the number of triangles of trussness k each edge is in.
- Time is proportional to the updated number of triangles.
- Propagation can cause $O(\mathcal{T}_G)$ time per update.
- Very good amortized time complexity $O(t(G) \cdot \mathcal{T}_G)$

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Idea

- List all minimal k-trusses
- Find a minimum hypergraph transversal

Max-Truss Breaking Heuristic (MBH)

- Let k' be the highest number such that G contains a k'-truss.
- Let M be the maximal k'-truss.
- Let $|\mathrm{TRI}_{\geq k'}(M, e)|$ be the number of triangles in M containing edge e.

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- Let k' be the highest number such that G contains a k'-truss.
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- Let $|\mathrm{TRI}_{\geq k'}(M, e)|$ be the number of triangles in M containing edge e.

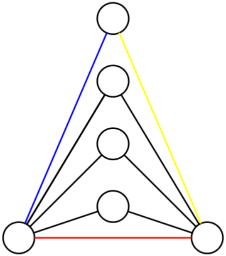
While $k' \geq k$:

MBH_S Delete the edge with the highest $|\mathrm{TRI}_{\geq k}(M, e)|$

MBH_C Delete the edge in the highest $|\mathrm{TRI}_{\geq k}(M, e)|/|\mathrm{TRI}_{<k}(M, e)|$

Save the Neighbours Heuristic (SNH)

- Give more weight to edges which are in many triangles
- Give lower weight to triangles whose edges have high support



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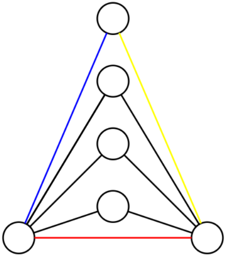
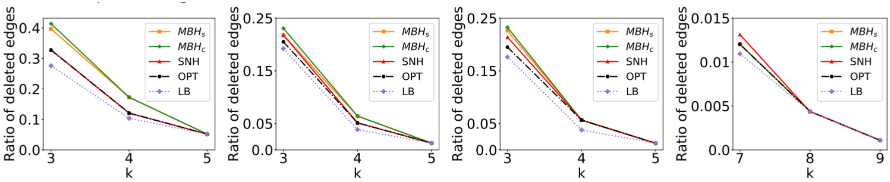


Delete edge with minimal

$$\sum_{\{e,f,g\}\in \mathrm{TRI}_{\geq k}} \frac{|\mathrm{TRI}_{\geq k}(M,e)|}{\max(|\mathrm{TRI}_{\geq k}(M,f)|-k+2,1)}+\frac{|\mathrm{TRI}_{\geq k}(M,e)|}{\max(|\mathrm{TRI}_{\geq k}(M,g)|-k+2,1)}$$

Small Networks



(a) TRIBES (b) KARATE (c) DOLPHINS (d) NETSCIENCE

Benchmarking Large Networks

Problem

How can we compare the accuracy of these heuristics on large data?

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Partition the graph into cliques.
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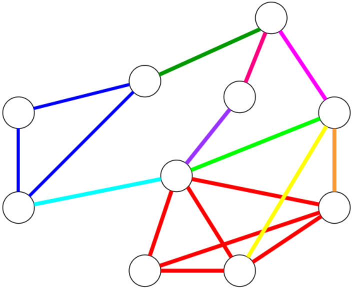

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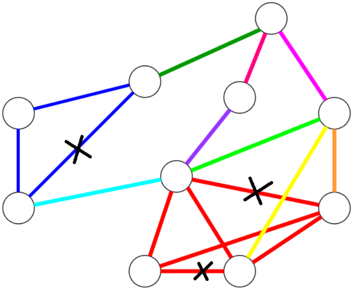

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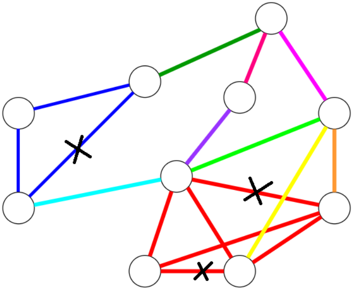

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Theorem (Turán)

A graph on n vertices and more than $\frac{(k-2)}{(k-1)} \cdot \frac{n^2}{2}$ edges contains a k-clique.

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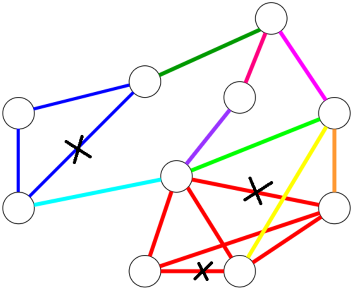

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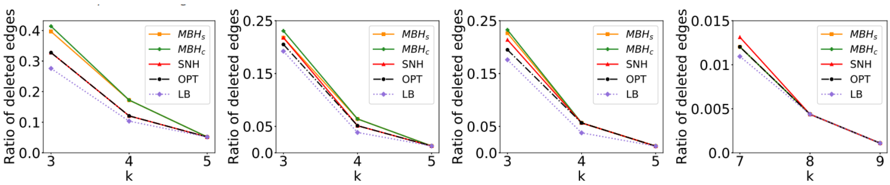
Theorem (Conte et al.)

A graph with m edges and $\mathcal{T}_G$ triangles has trussness at least $\mathcal{T}_G/m + 1$.

Small Networks



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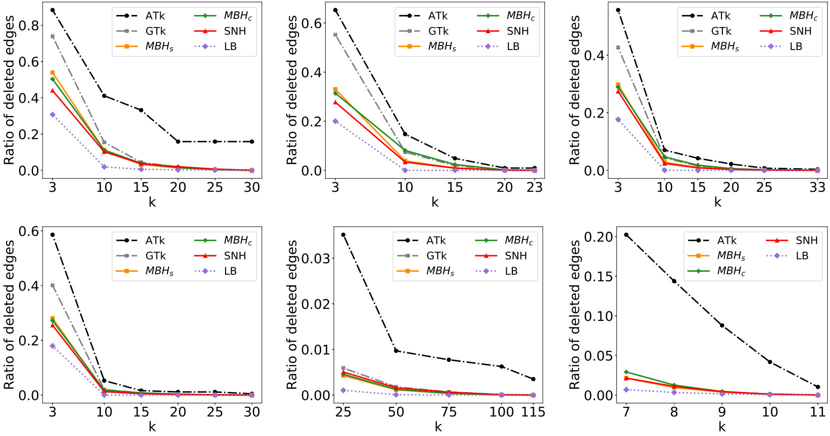

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