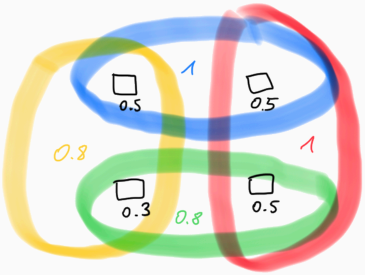
Decomposition of Probability Marginals for Security Games in Abstract Networks

Jannik Matuschke

KU Leuven

Input:

- ground set E
- set system $\mathcal{P} \subseteq 2^E$
- requirements $\pi \in [0,1]^{\mathcal{P}}$
- marginals $\rho \in [0,1]^E$



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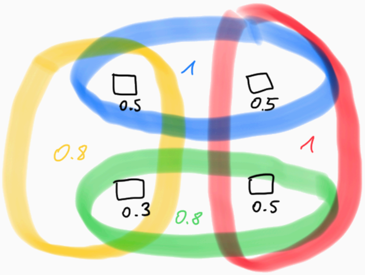

Goal: Find distribution for random set $S \subseteq E$ such that

$$\Pr\left[e \in S\right] = \rho_e \qquad \forall\, e \in E,$$
$$\Pr\left[P \cap S \neq \emptyset\right] \geq \pi_P \qquad \forall\, P \in \mathcal{P}.$$

Feasible Decompositions

We call a random set S with

$$\begin{aligned} \Pr\Big[e \in S\Big] \;&=\; \rho_e \qquad && \forall\, e \in E, \\ \Pr\Big[P \cap S \neq \emptyset\Big] \;&\geq\; \pi_P \qquad && \forall\, P \in \mathcal{P}, \end{aligned}$$

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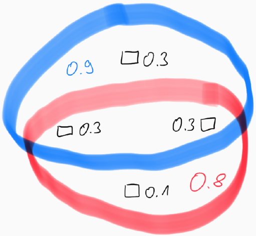

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A Necessary Condition

$$\sum_{e \in P} \rho_e \geq \pi_P \quad \forall\, P \in \mathcal{P} \quad (\star)$$

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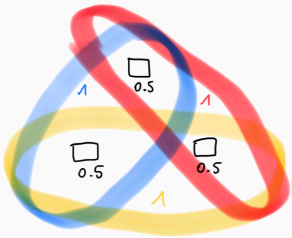

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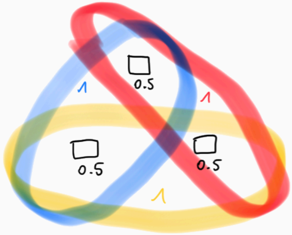

A system $(\mathcal{P}, \pi)$ is **$(\star)$-sufficient** if for all $\rho \in [0,1]^E$:

ρ is feasible. $\quad\Leftrightarrow\quad$ ρ fulfils $(\star)$.

Given: set system $(E, \mathcal{P})$,
costs $c \in \mathbb{R}_+^E$,
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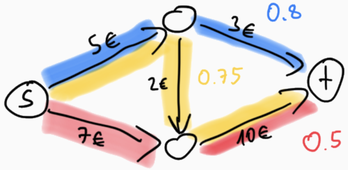

Motivation: Security Games

Given: set system $(E, \mathcal{P})$,
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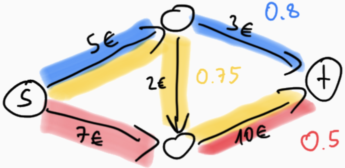
Defender

selects random set $S \subseteq E$ at cost $\sum_{e \in S} c_e$
wants to deter any attack at minimum cost



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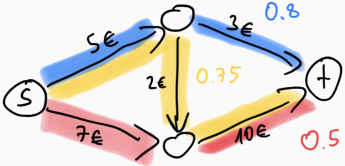

Attacker

selects $P \in \mathcal{P}$ or remains inactive

π_P: risk threshold for deterring attacker from P

Motivation: Security Games

Defender's problem:

$$
\begin{aligned}
\min \sum_{S \subseteq E} \sum_{e \in S} c_e\, x_S & \\
\text{s.t.} \sum_{S : P \cap S \neq \emptyset} x_S &\geq \pi_P \quad \forall\, P \in \mathcal{P} \\
\sum_{S \subseteq E} x_S &= 1 \\
x &\geq 0
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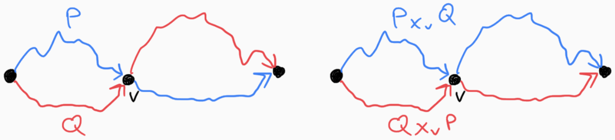
further applications: fairness/balance constraints in social choice, randomized algorithms

similar: Border's Theorem for auctions

$\mathcal{P} = \{s\text{-}t\text{-paths in a DAG}\}$, two settings for π:

(A) **Affine requirements:** $\pi_P = 1 - \sum_{e \in E} \mu_e$ for some $\mu \in [0,1]^E$

(C) **Conservation law:** $\pi_P + \pi_Q = \pi_{P \times_v Q} + \pi_{Q \times_v P}$ for $P, Q \in \mathcal{P}, v \in P \cap Q$



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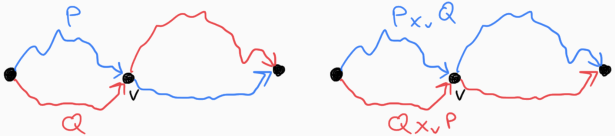

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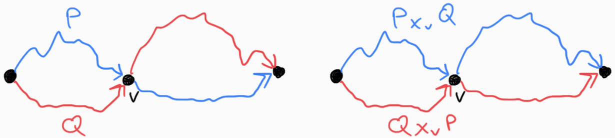

Their results:

- For (C): $(\mathcal{P}, \pi)$ is $(\star)$-sufficient.
- For (A): Decomposition can be computed efficiently.
- Consequence: Computation of Nash equilibria in interdiction game on DAG.

New Results

	DAGs (Dahan et al.)
Affine	efficient algorithm
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⊕ combinatorial shortest-path algorithm for abstract networks

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Also: NP-hard to decide feasibility of given ρ in general systems

Feasible Decompositions for Max-Flow/Min-Cut Systems

The Max-Flow/Min-Cut Property

$\mathcal{P}$ has the max-flow/min-cut property if

$$Q_{\mathcal{P}} := \left\{ \begin{array}{rlr} \sum_{e \in P} y_e & \geq 1 & \forall\, P \in \mathcal{P} \\ y_e & \geq 0 & \forall\, e \in E \end{array} \right\}$$ is integral.

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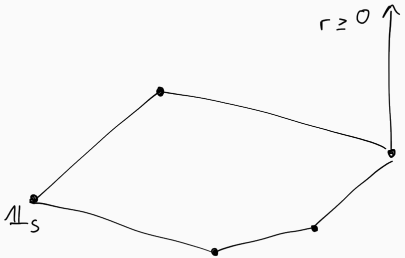

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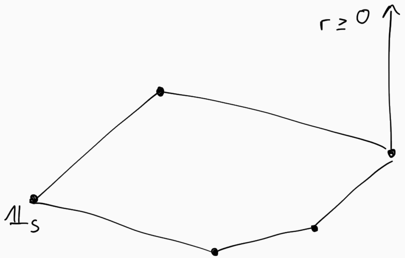

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Theorem. The following two statements are equivalent:

1. $\mathcal{P}$ has the MF/MC property.
2. $(\mathcal{P}, \pi)$ is $(\star)$-sufficient for all π fulfiling (A).

Proof: MF/MC implies $(\star)$-sufficiency for affine π

(A) $\pi_P = 1 - \sum_{e \in P} \mu_e$ for some $\mu \in [0,1]^E$

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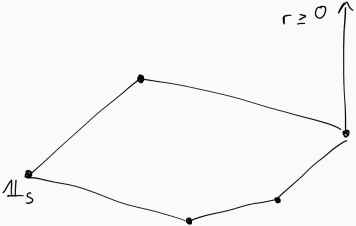

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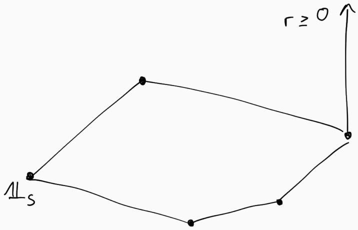

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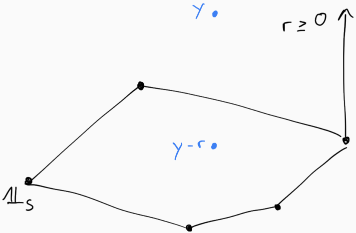

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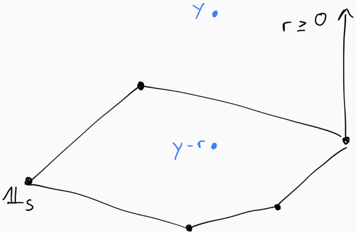

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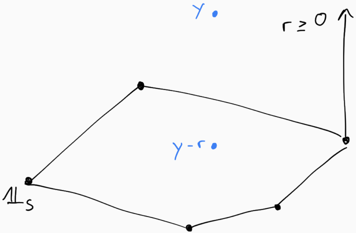

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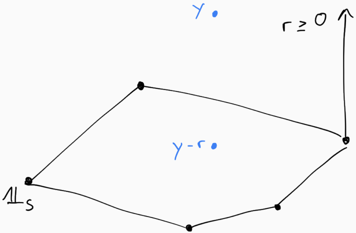

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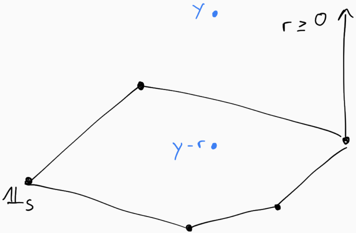

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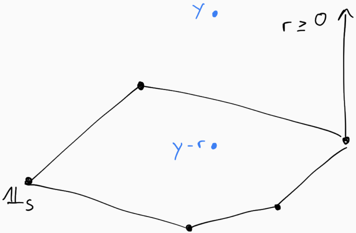

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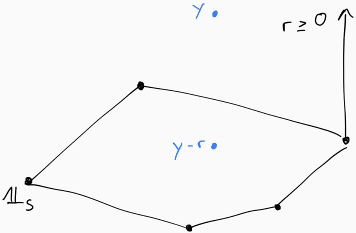

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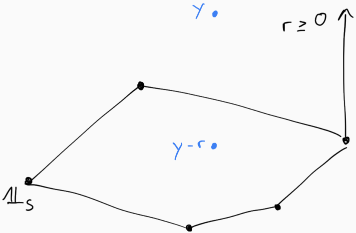

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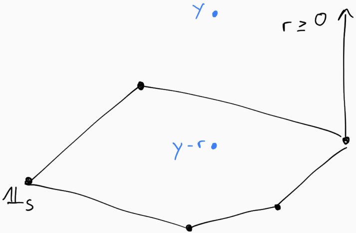

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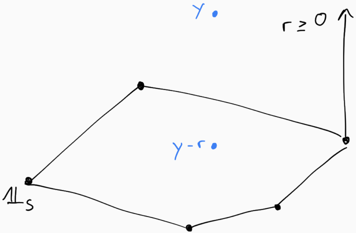

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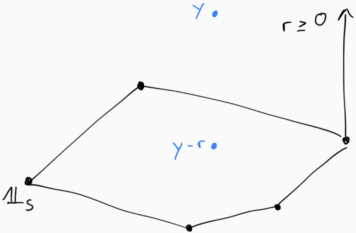

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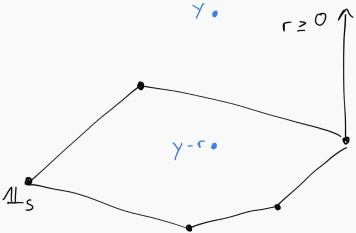

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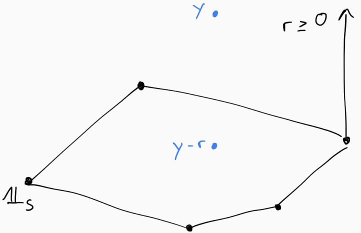

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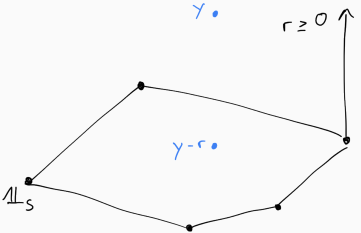

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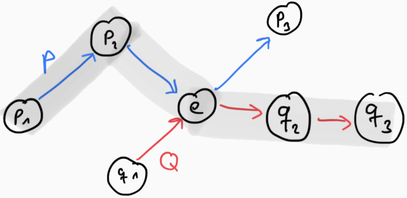
Note: We can construct R if we can solve SEPARATION for $Q_{\mathcal{P}}$.

Abstract Networks

An **abstract network** is a set system $(E, \mathcal{P})$ such that

- for every $P \in \mathcal{P}$, there is an order $\preceq_P$ on P,
- for every $P, Q \in \mathcal{P}$ and $e \in P \cap Q$, there is $P \times_e Q \in \mathcal{P}$ with

$$P \times_e Q \subseteq \{p \in P \ : \ p \preceq_P e\} \cup \{q \in Q \ : \ e \preceq q\}.$$

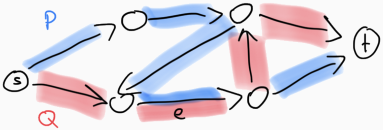


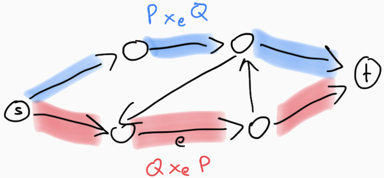
$$P \times_e Q = q_3 \to p_2 \to q_2$$

Example: s-t-Paths in a Digraph

Digraph $D = (V, A)$ $\quad\rightarrow\quad$ $E = A$ (or $E = V$, or $E = V \cup A$)

$\mathcal{P} = \{s\text{-}t\text{-paths in } D\}$

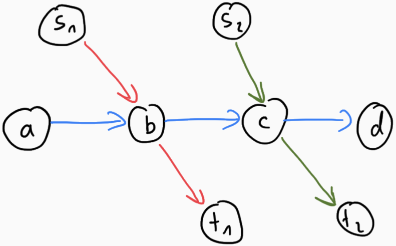




Another Example

$$E = \{a, b, c, d, s_1, s_2, t_1, t_2\}$$

$$\mathcal{P} = \{\{a, b, c, d\},\ \{s_1, b, t_1\},\ \{s_2, c, t_3\},$$
$$\{s_1, d\},\ \{s_2, d\},\ \{a, t_1\},\ \{a, t_2\}\}$$



Note: No path starting with s_1 and ending with t_2.

Abstract Networks and Max-Flow/Min-Cut

Abstract Max Flow

$$\begin{aligned} \max \sum_{P \in \mathcal{P}} x_P & \\ \text{s.t.} \sum_{P : e \in P} x_P &\leq u_e \quad \forall\, e \in E \\ x &\geq 0 \end{aligned}$$

Abstract Min Cut

$$\begin{aligned} \min \sum_{e \in E} u_e y_e & \\ \text{s.t.} \quad \sum_{e \in P} y_e &\geq 1 \quad \forall\, P \in \mathcal{P} \\ y &\geq 0 \end{aligned}$$

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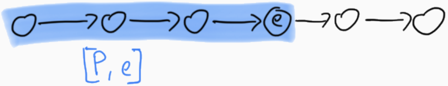
Martens & McCormick (IPCO 2008): combinatorial algorithm for weighted version

Feasible Decompositions in Abstract Networks

$$\alpha_e := \min_{P \in \mathcal{P}} \sum_{f \in [P,e]} \rho_f + \mu_f$$



Feasible Decompositions in Abstract Networks $(\pi_P = 1 - \sum_{e \in P} \mu_e)$

$$S_\tau := \left\{ e \in E \,:\, \alpha_e - \rho_e \leq \tau \leq \alpha_e \right\}$$

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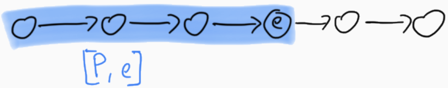

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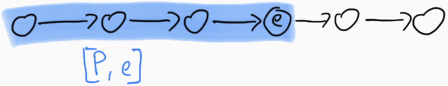

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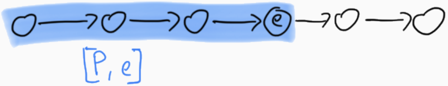

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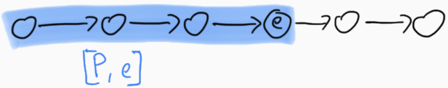

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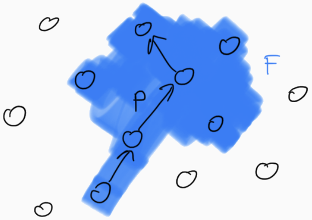
By $(\star)$: $\alpha_t \ge 1$ for last element t of P

Shortest Paths in Abstract Networks

How Do We Access Abstract Networks?

Membership oracle for an abstract network:
Given $F \subseteq E$, either

- return $P \in \mathcal{P}$ (and $\preceq_P$) with $P \subseteq F$,
- or assert that no such P exists.



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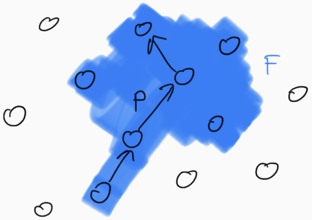

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McCormick's Max Abstract Flow algorithm uses membership oracle.

Question (McCormick 1996): Can a stronger oracle (e.g., shortest paths) yield a strongly poly-time algorithm for Max Abstract Flow?

Shortest Paths in Abstract Networks

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Task: find $P \in \mathcal{P}$ minmizing $c(P) := \sum_{e \in P} c_e$

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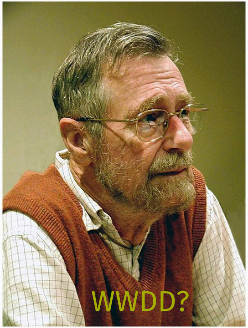

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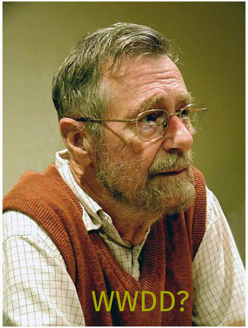
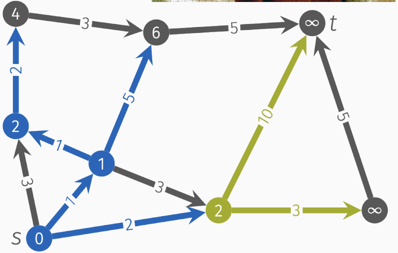


Dijkstra's Algorithm

- initialize:
  - label ϕ_v for $v \in V$
  - s-v-path Q_v with $\phi_v = c(Q_v)$
  - set of processed nodes T
- while $\min_{v \in V \setminus T} \phi_v < \phi_t$

  pick $v \in \operatorname{argmin}_{w \in V \setminus T} \phi_w$

  $\texttt{process}(v)$



Shortest Paths in Abstract Networks

Given: abstract network $(E, \mathcal{P})$, costs $c \in \mathbb{R}_+^E$

Task: find $P \in \mathcal{P}$ minmizing $c(P) := \sum_{e \in P} c_e$

Dijkstra's Algorithm (adapted)

- initialize:
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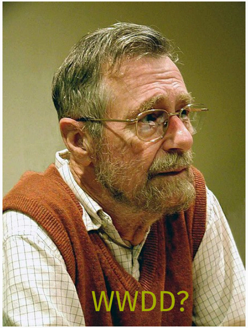

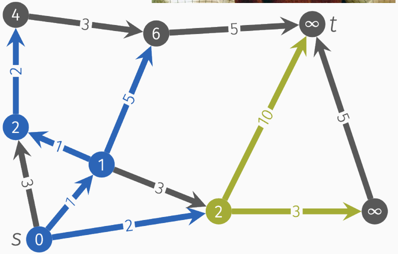

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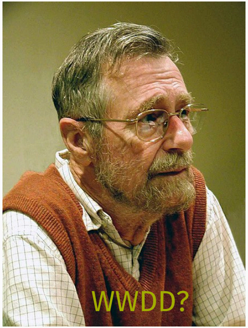

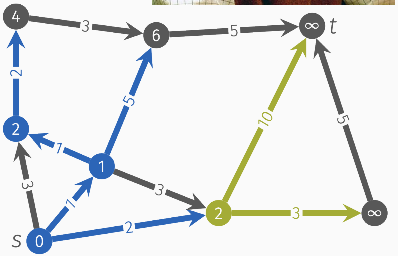

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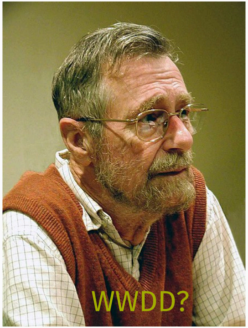
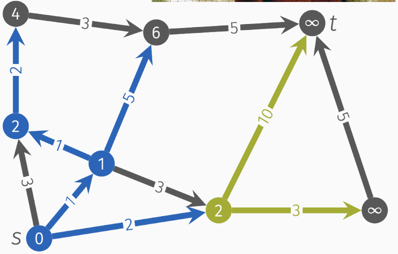
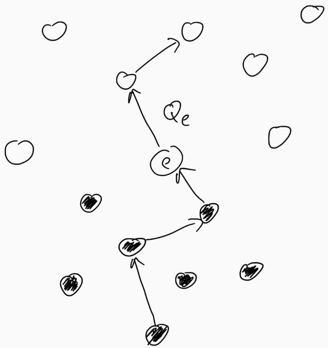




Processing Elements

How to find all relevant ways to continue $[Q_e, e]$?



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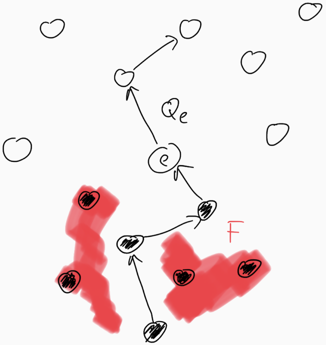

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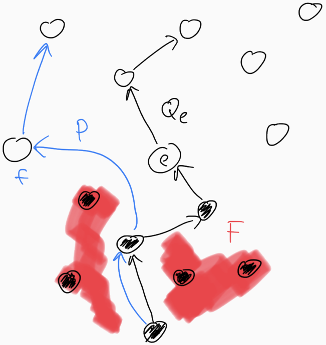

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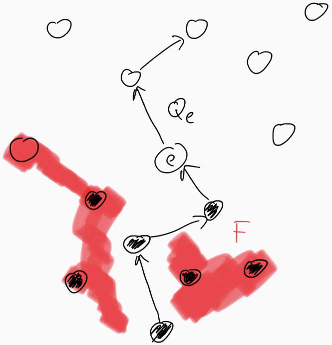

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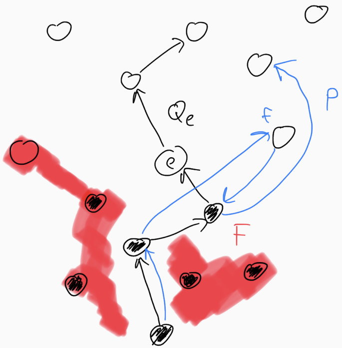

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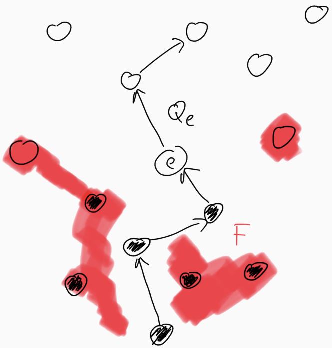

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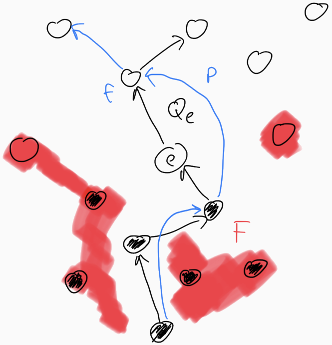

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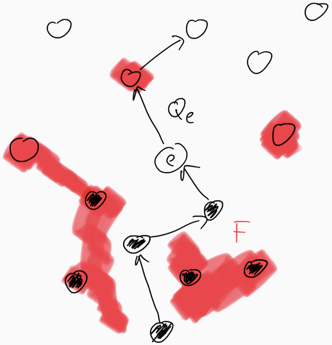

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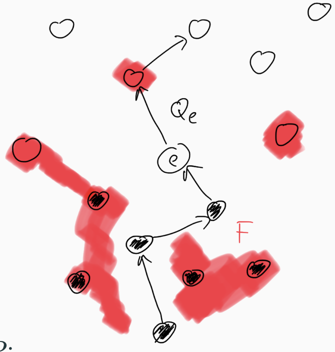

Lemma. After $\texttt{process}(e)$, for every $P \in \mathcal{P}$ with $e \in P$:

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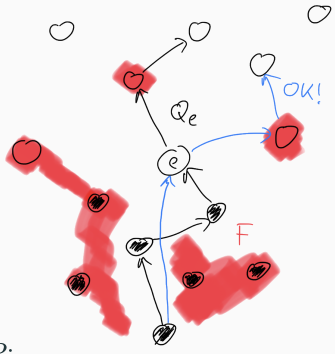

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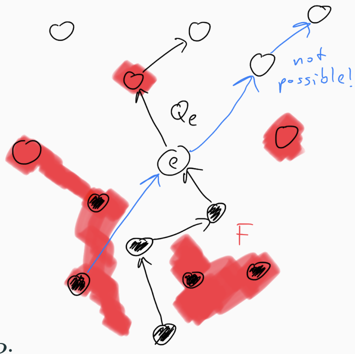

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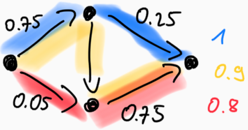
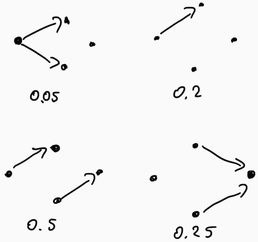
Conclusion





- $(\star)$-**sufficiency** allows formulating problems via their marginals:

$$\sum_{e \in P} \rho_e \geq \pi_P \quad \forall\, P \in \mathcal{P} \quad (\star)$$

- many systems are $(\star)$-sufficient, including **abstract networks**
- feasible decompositions can be computed via a **shortest-path algorithm**

Overview & Open Questions

	DAGs (Dahan et al.)	Abstract Networks (incl. digraphs w. cycles)	Max-Flow/Min-Cut
Affine	efficient algorithm	efficient algorithm (explicit description) $\oplus$	characterize $(\star)$-sufficiency
	$\Updownarrow$		
Conservation	$(\star)$-sufficient (exp.-time algorithm)	$(\star)$-sufficient (oracle-poly)	TDI systems?

$\oplus$ combinatorial shortest-path algorithm for abstract networks

Strongly poly-time algorithm for Abstract Max Flow?

Also: NP-hard to decide feasibility of given ρ in general systems

Poly-time algorithms for some non-$(\star)$-sufficient systems?

$(\star)$-sufficiency under additional constraints on decomposition?

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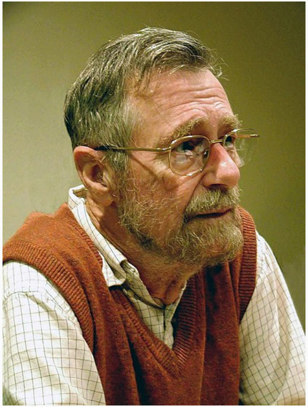

“Portrait of Edsger W. Dijkstra, one of the greatest mathematicans in history of modern mathematics.”