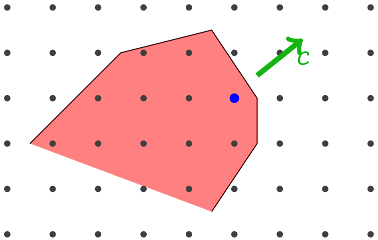
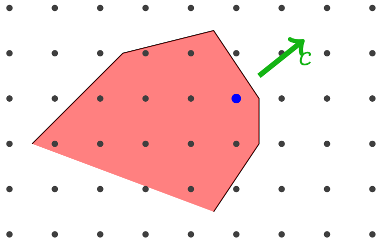
# On the complexity of integer programming




Friedrich Eisenbrand

# Integer Programming

$$\max\{c^T x : x \in \mathbb{Z}^n, Ax \leq b\}$$




*NP-complete* *versatile optimization problem.*
*Example: SAT*

$\phi$ - $\{x_2, \bar{x_4}, x_7\}, \{x_3, \bar{x_4}, x_5\}, \ldots$

$$\begin{aligned} x_2 + (1 - x_4) + x_7 &\geq 1 \\ x_3 + (1 - x_4) + x_5 &\geq 1 \\ &\vdots \end{aligned}$$

$x \geq 0, \quad x \in \mathbb{Z}^n$

*Other examples: Scheduling, TSP, Cryptography, etc.*

$\phi$ is sat. $\iff$ $Ax \leq b$, $x \in \mathbb{Z}^n$ is feasible

# Integer Programming

$$\begin{array}{ccc} \max & c^T x & \\ A\,x & = & b \\ x & \geq & 0 \\ x & \in & \mathbb{Z}^n \end{array}$$

*Standard Form*

$$\begin{array}{ccc} \max & c^T x & \\ A\,x & \leq & b \\ x & \in & \mathbb{Z}^n \end{array}$$

*Inequality Form*

$$(m \cdot \Delta)^{O(m^2)}$$

*(Papadimitriou 1981)*

# Integer Programming

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$A \in \mathbb{Z}^{\mathbf{m \times n}}$
$b \in \mathbb{Z}^{\mathbf{m}}$
$||A||_\infty \leq \Delta$
$||b||_\infty \leq \Delta$

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n = # VAR:

*Standard Form*

$(m \cdot \Delta)^{O(m^2)}$

(Papadimitriou 1981)

*Inequality Form*

Lenstra ~~18~~ 1983 $2^{O(n^2)}$

KANNAN 1986 $n^{O(n)}$

~~BDI~~ DADUSH best ~~of~~ constants.

**Here:**

$(m \cdot \Delta)^{O(m)} \cdot ||b||_\infty^2$ upper bound (E. & Weismantel 2018/19), (Jansen Rohwedder 2018/19)

Matching lower bound even for 0/1-matrices (Knop, Pilipczuk, Wrochna 2020)

# $(m \cdot \Delta)^{O(m^2)}$ — Papadimitriou 1981

$$\begin{array}{ccc} \max & c^T x & \\ A\,x & = & b \\ x & \geq & 0 \\ x & \in & \mathbb{Z}^n \end{array}$$

*Fact:* *There exists optimal solution $x^*$ with*

$$\|x^*\|_\infty \leq U = (m \cdot \Delta)^m$$

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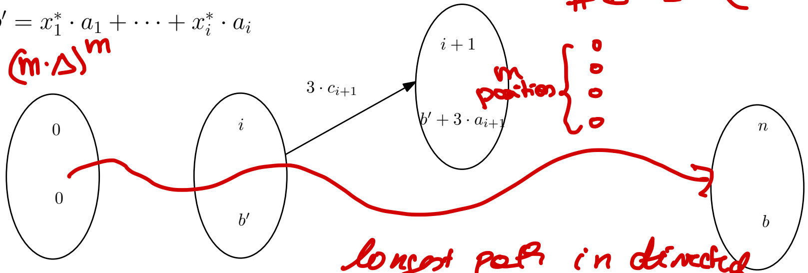
$\# b' \leq (m \cdot \Delta)^{m^2}$

$$b' = x_1^* \cdot a_1 + \cdots + x_i^* \cdot a_i$$

$\|b'\|_\infty \approx (m \cdot \Delta)^m$




longest path in directed acyclic graph.

Number of Nodes approximately $(m \cdot \Delta)^{m^2}$

# $(m \cdot \Delta)^{O(m^2)}$ — Papadimitriou 1981

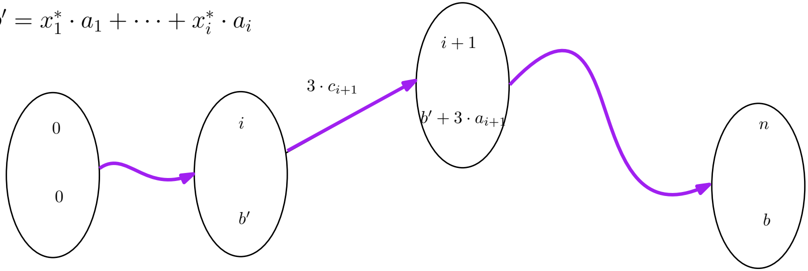
$$\begin{array}{lll} \max & c^T x & \\ A\,x & = & b \\ x & \geq & 0 \\ x & \in & \mathbb{Z}^n \end{array}$$

$\cdot \, \Delta^m$

*Fact:* *There exists optimal solution $x^*$ with*

$$\|x^*\|_\infty \leq U = (m \cdot \Delta)^m$$

$$b' = x_1^* \cdot a_1 + \cdots + x_i^* \cdot a_i$$

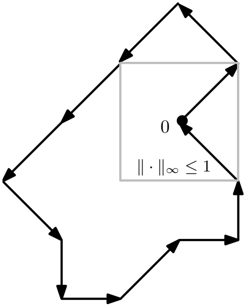



Number of Nodes approximately $(m \cdot \Delta)^{m^2}$ $\rightsquigarrow (m \cdot \Delta)^m$

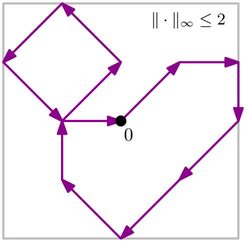
# The Steinitz Lemma (Steinitz 1913)

Let $x_1, \ldots x_n \in \mathbb{R}^m$ such that $||x_i|| \leq 1$ for each $i$ and $\sum_i x_i = 0$, then there exists a permutation $\pi$ such that

$$||\sum_{i=1}^{k} x_{\pi_i}|| \leq m \text{ for each } k.$$




$m$



# $(m \cdot \Delta)^{O(m)}$ — A smaller Steinitz state space

$$\begin{array}{lll} \max & c^T x & \\ A\,x & = & b \\ x & \geq & 0 \\ x & \in & \mathbb{Z}^n \end{array}$$

**Assumption:**
$||A||_\infty, ||b||_\infty \leq \Delta$

- Let $x^*$ be optimal solution

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- Let $x^*$ be optimal solution
- **Steinitz sequence**:

$$0 = \underbrace{a_1 + \cdots + a_1}_{x_1^* \text{ times}} + \quad \cdots \quad + \underbrace{a_n + \cdots + a_n}_{x_n^* \text{ times}} \quad - \quad b$$

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- Can be re-arranged such that partial sums of columns have $|| \cdot ||_\infty \leq 2\,m \cdot \Delta$

# $(m \cdot \Delta)^{O(m)}$ — A smaller Steinitz state space

$$\begin{array}{llll} \max & c^T x & & \\ & A\,x & = & b \\ & x & \geq & 0 \\ & x & \in & \mathbb{Z}^n \end{array}$$

$y \cdot x =$ col. of A.

**Assumption:**
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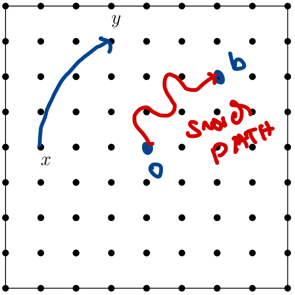


$2\,m \cdot \Delta$

- Let $x^*$ be optimal solution
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~~2\~~ $(m \cdot \Delta)^m$ many nodes:

$y - x = a_i$

## $(m \cdot \Delta)^{O(m)}$ — A smaller Steinitz state space

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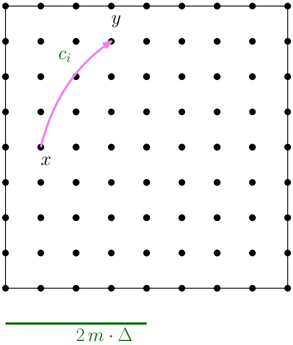


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$y - x = a_i$

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$$
\begin{array}{lcl}
\max & c^T x & \\
A\,x & = & b \\
x & \geq & 0 \\
x & \in & \mathbb{Z}^n
\end{array}
$$

**Assumption:**
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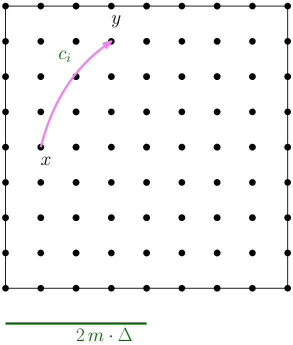


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- Can be re-arranged such that partial sums of columns have $||\cdot||_\infty \leq 2\,m \cdot \Delta$
- $(m \cdot \Delta)^{O(m)}$ nodes

Feasibility problem in $(m \cdot \Delta)^{O(m)}$ time.

$y - x = a_i$

# $(m \cdot \Delta)^{O(m)} \cdot ||b||$ — A milder dependence on $b$

$$\begin{array}{lcc} \max & c^T x & \\ A\,x & = & b \\ x & \geq & 0 \\ x & \in & \mathbb{Z}^n \end{array}$$

**Assumption: Only** $||A||_\infty \leq \Delta$

- Let $x^*$ be optimal solution with $k = ||x^*||_1$

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$$0 = \underbrace{a_1 - b/k + \cdots + a_1 - b/k}_{x_1^* \text{ times}} + \quad \cdots \quad + \underbrace{a_n - b/k + \cdots + a_n - b/k}_{x_n^* \text{ times}}$$

## $(m \cdot \Delta)^{O(m)} \cdot ||b||$ — A milder dependence on $b$

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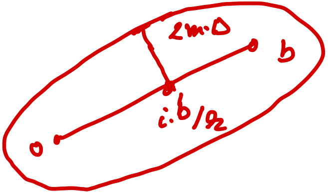

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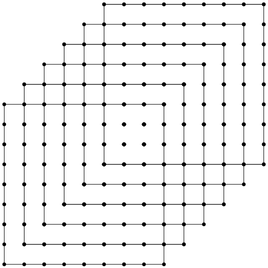

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$$(m \cdot \Delta)^{O(m)} \cdot ||b||$$

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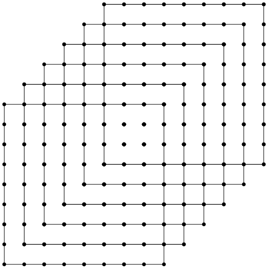

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- $(m \cdot \Delta)^{O(m)} \cdot ||b||$ nodes

# $(m \cdot \Delta)^{O(m)} \cdot ||b||$ — A milder dependence on $b$

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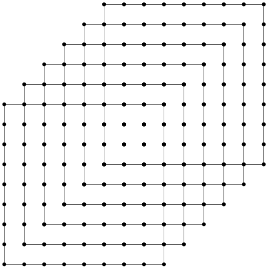

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- Can be re-arranged such that partial sums of columns have distance $||\cdot||_\infty \leq 2\,m \cdot \Delta$ from line-segment $0, b$

- $(m \cdot \Delta)^{O(m)} \cdot ||b||$ nodes

*Theorem: IP can be solved in time $(m \cdot \Delta)^{O(m)} \cdot ||b||$.*

# Lower bounds based on ETH

**Exponential Time Hypothesis (ETH)**: There exists $\epsilon > 0$ such that 3-SAT cannot be solved in time $2^{\epsilon \cdot n}$.

*Theorem:* *For $0/1$-matrices $A$, there is no* $2^{o(m \log m)}(n + ||b||_\infty)^{o(m)}$ *algorithm for IP, unless ETH is false.*

*(Knop, Pilipczuk, & Wrochna 2019)*

$(m \cdot \Delta)^{O(m)}$

$\Delta = ||A||_\infty$

$0/1 : \Delta = 1$

# Sparsification Lemma

Impagliazzo, Paturi & Zane 2001

$\varepsilon > 0$ , $\Phi$ k-SAT FORMULA

SPARSIFICATION

$\Phi_1 \vee \dots \vee \Phi_{2^{\varepsilon \cdot n}}$

$\Phi_i$: each variable appears in at most $c(k, \varepsilon)$ clauses

ETH $\Rightarrow$ $\exists\, \varepsilon > 0$ s.th. 3-SAT, each variable appears in constant many clauses cannot be solved in $2^{\varepsilon \cdot n}$

3 SAT. each var. appears in $\leq 4$ clauses

$\Phi$

$\Downarrow$

ETH

$\exists\, \varepsilon > 0$ st. IP cannot be solved in $2^{\varepsilon \cdot n}$ $\Rightarrow$ $Ax = b$

$x \geq 0$ , $x \in \mathbb{Z}^n$

$O(n) \times O(n)$

$A \in \{0, \pm 1\}$

$x_i + z_i = 1$

$x_i, z_i \geq 0$

$s_i + \bar{s}_i = 1$

$\{x_1, \bar{x}_2, x_3\}$

$\Downarrow$

$x_1 + z_2 + x_3 - s_1 - s_2 = 1$

$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ $\leftarrow$ constant number of nonzero entries.

$$\log(n)\left\{\begin{bmatrix} A_1 \\ \hline \\ \hline A_{n/\log(n)} \end{bmatrix}\right. x = \begin{bmatrix} b_1 \\ \vdots \\ b_{n/\log n} \end{bmatrix}$$

<u>Aggregation:</u> $S = \{x \in \{0,1\}^n : \quad A_i x = b_i\}$

$\cap\!|$

$S' = \{x \in \{0,1\}^n : \quad \lambda^T A_i x = \lambda^T b_i\}$

$\lambda \in \{0,\dots,N\}^{\log(n)}$ at random.

$x^* \in S' \setminus S \iff \lambda^T \underbrace{(A_i x^* - b_i)}_{=b'} = 0$ $\qquad \lambda^T b' = (1, \sim)$

$\|b'\|_\infty \leq 6$ $\qquad \lambda_1 \cdot 1 + \sim$

$\# \text{ of } b' \leq 6^{\log(m)}$

$= m^{\log(6)}$

$\|\lambda\| \leq N = m^{20}$ at random

$\Pr(S \neq S') \leq \frac{1}{m^{10}}$

$Ax = b$
$x \geq 0$ $\quad \Rightarrow \quad A'x = b' \qquad A' \in \mathbb{Z}^{\frac{n}{\log(n)} \times n}$

$\Delta = n^{20}$

IP running time: $n^{O\left(\frac{n}{\log(n)}\right)} = 2^{O(n)}$

Lower bound: $2^{\Omega(n)}$ (ETH)

# Proximity bounds

$$\begin{array}{lll} \max & c^T x & \\ A\,x & \leq & b \\ & x \in \mathbb{Z}^n & \end{array}$$

$x^*$ opt. sol of LP relaxation

There exists integer optimal solution $z^*$ with

$$\|z^* - x^*\|_\infty \leq n^2 \Delta^n$$

(Cook et al. 1986)

# Proximity bounds

$$\begin{array}{ll} \max & c^T x \\ A\,x & = \quad b \\ & u \geq x \geq 0 \\ & x \in \mathbb{Z}^n \end{array}$$

$x^*$ opt. sol of LP relaxation

There exists integer optimal solution $z^*$ with

$$\|z^* - x^*\|_1 \leq m \cdot (2\,m \cdot \Delta + 1)^m$$

generalization of a result of Aliev, Henk & Oertel (2017)

# IP with upper bounds on the variables

$$\begin{aligned} \max \quad & c^T x \\ A\,x \quad &= \quad b \\ 0 \leq x &\leq u \\ x &\in \mathbb{Z}^n \end{aligned}$$

Consider $x \in \{0,1\}^n$ and $A \in \{0,1\}^{m \times n}$

$x^*$ LP-OPT

$$\begin{aligned} &\max c^T(z - \lfloor x^* \rfloor) \\ &A\,(z - \lfloor x^* \rfloor) = A\{x^*\} \\ &(z - \lfloor x^* \rfloor) \in \{0, \pm 1\}^n \\ &\|z - \lfloor x^* \rfloor\|_1 \leq m^{O(m)} \end{aligned}$$

$A \cdot x^* = b$

$A \cdot z = b$

$A(z - x^*) = 0$

$A \cdot (z - \lfloor x^* \rfloor) = A \cdot \{x^*\}$

$m^{O(m^2)}$

$m$ [illegible]

$(m \cdot \Delta)^{m^2}$

(E. & Weismantel 2018)

## Open problem

Is there an $(m\Delta)^{O(m)}$ algorithm for IP with upper bounds on the variables?