Relaxation Complexity: Algorithmic Possibilities and Limitations

joint work with Manuel Aprile, Gennadiy Averkov, Marco Di Summa, and Matthias Schymura

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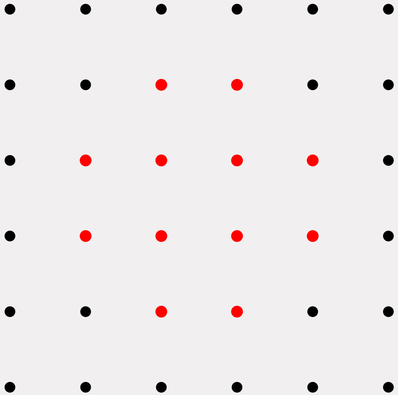
Dutch Optimization Seminar, February 23, 2023

Introduction

Main Goal: find "good" linear representations of integer points *X* in polytopes *P*

Motivation: integer programming formulations

What does "good" mean?



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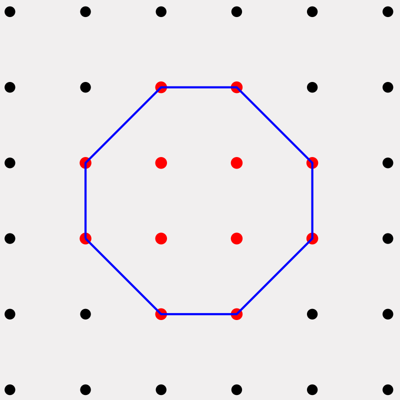

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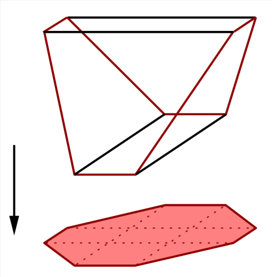

source: *Extended Formulations for Polygons*, Fiorini, Rothvoß, Tiwary. Discr. & Comp. Geom. 48, 2012

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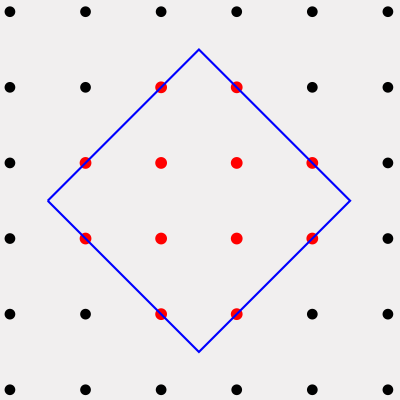
- extended formulations: allow additional variables for description
- relaxation complexity: minimal number of linear constraints



Relaxation Complexity—Definition

Let $P \subseteq \mathbb{R}^d$ be a polytope and let

$$X = P \cap \mathbb{Z}^d.$$

Any such X is called lattice-convex, because

$$\mathrm{conv}(X) \cap \mathbb{Z}^d = X.$$

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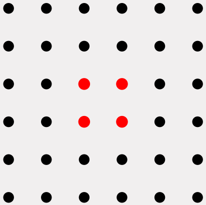
Definition (Weltge 2015, Kaibel & Weltge 2015)

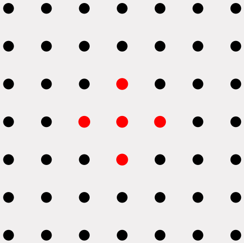
Any polyhedron $Q \subseteq \mathbb{R}^d$ with $X = Q \cap \mathbb{Z}^d$ is called a relaxation of X.
The relaxation complexity $\mathrm{rc}(X)$ is the minimal number of facets of a relaxation of X.

Relaxation Complexity—Examples

0/1 cube



cross-polytope



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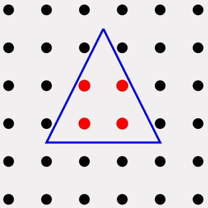

$\mathrm{rc}(X) = d + 1$

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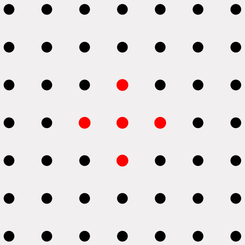

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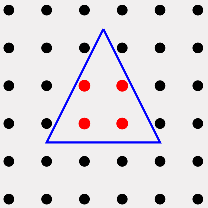

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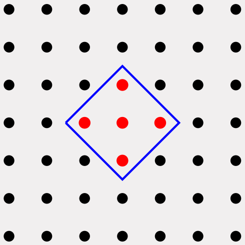


$$\text{rc}(X) = \begin{cases} 4, & \text{if } d = 2, \\ d+1, & \text{otherwise} \end{cases}$$

TU/e

Relaxation Complexity—Basic Properties

Let $P \subseteq \mathbb{R}^d$ be a polytope and let $X = P \cap \mathbb{Z}^d$

Definition (Weltge 2015, Kaibel & Weltge 2015)

Any polyhedron $Q \subseteq \mathbb{R}^d$ with $X = Q \cap \mathbb{Z}^d$ is called a relaxation of X.
The relaxation complexity $\mathrm{rc}(X)$ is the minimal number of facets of a relaxation of X.

- $\mathrm{rc}(X)$ is the minimal number of inequalities needed to separate X from $\mathbb{Z}^d \setminus X$ (within $\mathrm{aff}(X)$)
- more generally, for $Y \subseteq \mathbb{Z}^d$, the minimal number of inequalities needed to separate X from $Y \setminus X$ is $\mathrm{rc}(X, Y) \Rightarrow \mathrm{rc}(X, Y) \leq \mathrm{rc}(X)$
- $\mathrm{rc}_{\mathbb{Q}}(X)$ and $\mathrm{rc}_{\mathbb{Q}}(X, Y)$ mean restricting to rational polyhedra $\Rightarrow \mathrm{rc}(X) \leq \mathrm{rc}_{\mathbb{Q}}(X)$
- $\mathrm{rc}(X) \leq \#$facets of $\mathrm{conv}(X)$

Fundamental Questions

Let $X \subseteq \mathbb{Z}^d$ be lattice-convex.

Question

Are $\mathrm{rc}(X)$ and $\mathrm{rc}_{\mathbb{Q}}(X)$, and corresponding minimal relaxations, computable?

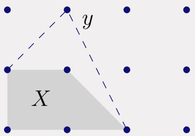
Let $\Delta_d = \{0, e^1, \dots, e^d\}$.

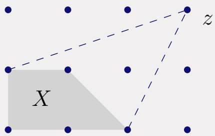
Question (Kaibel & Weltge 2015)

Does $\mathrm{rc}(X) = \mathrm{rc}_{\mathbb{Q}}(X)$ hold? In particular, is $\mathrm{rc}(\Delta_d) = \mathrm{rc}_{\mathbb{Q}}(\Delta_d)$ true?

Outline

Main Tool—Observers





Definition (Observers)

Let $X \subseteq \mathbb{Z}^d$ be lattice-convex. A point $y \in \mathbb{Z}^d \setminus X$ is called an observer of X, if $\text{conv}(X \cup \{y\}) \cap \mathbb{Z}^d = X \cup \{y\}$, that is, $X \cup \{y\}$ is also lattice-convex.
We write

$$\text{Obs}(X) := \left\{ y \in \mathbb{Z}^d \setminus X \ : \ y \text{ is an observer of } X \right\}.$$

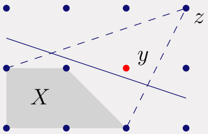
Observers

The observers are certifying the separation of X from $\mathbb{Z}^d \setminus X$!

Let $X \subseteq \mathbb{Z}^d$ be lattice-convex and let $Ax \leq b$ be a system of linear inequalities. The following are equivalent:

- The system $Ax \leq b$ separates X from $\mathbb{Z}^d \setminus X$.
- The system $Ax \leq b$ separates X from $\text{Obs}(X)$.

$\implies \text{rc}(X) = \text{rc}(X, \text{Obs}(X))$



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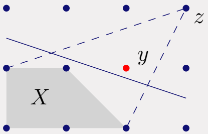

Observation on Computability (Averkov & Schymura 2021)

If $\mathrm{Obs}(X)$ is finite (and computable), then deciding Is $\mathrm{rc}(X) \leq k$? reduces to solving a MIP with binary integer variables.

Finitely Many Observers

Theorem (Averkov & Schymura 2021)

Let $X \subseteq \mathbb{Z}^d$ be lattice convex and full-dimensional. If

1. X is parity-complete, or
2. $\operatorname{conv}(X)$ contains an interior lattice point, or
3. the lattice width $w(X)$ of X satisfies $w(X) > w^\infty(d)$, where $w^\infty(d)$ is the so-called finiteness threshold width,

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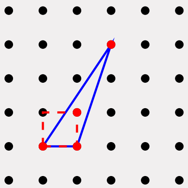

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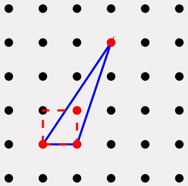

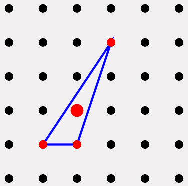

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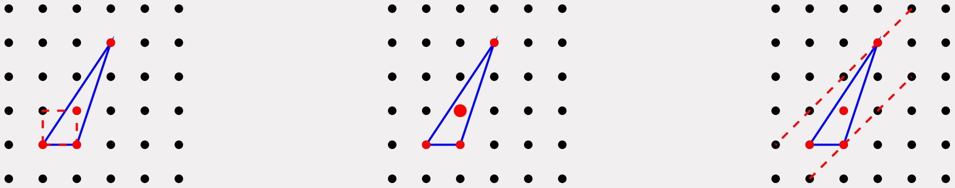


Computability in Dimension 2

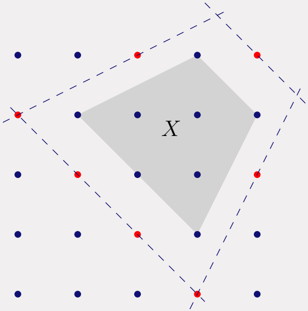
Observers for $d = 2$

- let $X \subseteq \mathbb{Z}^2$ be finite and lattice convex with $\text{conv}(X) = \{x \in \mathbb{R}^2 \; : \; a_i x_1 + b_i x_2 \leq c_i, \; i \in [m]\}$
- for each $i \in [m]$, assume a_i and b_i are co-prime

Proposition (Weltge 2015)

If $X \subseteq \mathbb{Z}^2$ is full-dimensional, finite, and lattice-convex, $\text{Obs}(X)$ are the lattice points on the boundary of

$$\{x \in \mathbb{R}^2 \; : \; a_i x_1 + b_i x_2 \leq c_i + 1, \; i \in [m]\}.$$



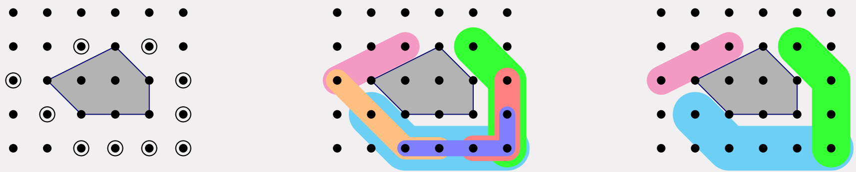
Finding Relaxation Complexity for $d = 2$

Theorem (Averkov, H., & Schymura 2021)

Let $V \subseteq \mathbb{Z}^2$ be finite and 2-dimensional, let $X = \text{conv}(V) \cap \mathbb{Z}^2$, and $Y = \text{Obs}(X)$. Then,

1. $\text{Obs}(X)$ can be computed in $O(|V| \log|V| + |Y| + \gamma|V|)$ time;
2. $\text{rc}(X)$ can be computed in $O(|V| \log|V| + |V||Y| \log|Y| + \gamma|V|)$ time,

where γ is an upper bound on the binary encoding size of any point in V.



Computable Bounds

Strategy

$$\mathrm{rc}(X) \leq \mathrm{rc}_{\mathbb{Q}}(X)$$

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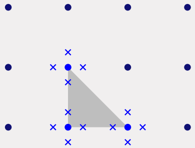
Computable Upper Bounds

Approach Give up strict separation of X from $\mathbb{Z}^d \setminus X$ for the sake of robustness.



- $X \subseteq \mathbb{Z}^d$ full-dimensional and lattice-convex
- for $\varepsilon > 0$, let $X_\varepsilon := X + \mathcal{B}^1_\varepsilon$,
  where $\mathcal{B}^1_\varepsilon = \{0, \pm\varepsilon \boldsymbol{e}_1, \dots, \pm\varepsilon \boldsymbol{e}_d\}$

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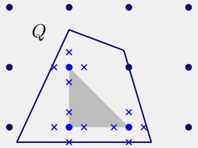

Definition (ε-relaxation complexity)

A polyhedron $Q \subseteq \mathbb{R}^d$ is an ε-relaxation of X, if $X_\varepsilon \subseteq Q$ and $X = \text{int}(Q) \cap \mathbb{Z}^d$.
We define $\text{rc}_\varepsilon(X)$ as the smallest number of facets of an ε-relaxation of X.

- $\varepsilon \geq \varepsilon' \implies \text{rc}_\varepsilon(X) \geq \text{rc}_{\varepsilon'}(X)$, since $\text{conv}(X_{\varepsilon'}) \subseteq \text{conv}(X_\varepsilon)$

Computable Upper Bounds

Theorem (Averkov, H., & Schymura 2021)

1. For any $\varepsilon > 0$, there is a computable finite set $Y^\varepsilon \subseteq \mathbb{Z}^d$ with $\mathrm{rc}_\varepsilon(X) = \mathrm{rc}_\varepsilon(X, Y^\varepsilon)$.
2. If $\varepsilon > 0$ is rational, then $\mathrm{rc}_\varepsilon(X)$ can be computed in finite time.
3. $\mathrm{rc}_\mathbb{Q}(X) = \mathrm{rc}_0(X) := \min_{\varepsilon > 0} \mathrm{rc}_\varepsilon(X)$

Consequences

$\rightsquigarrow$ finite algorithm to compute (eventually tight) upper bounds on $\mathrm{rc}_\mathbb{Q}(X)$

$\rightsquigarrow$ we just don’t know when to stop

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Key ideas for the proof

2. computing $\mathrm{rc}_\varepsilon(X, Y^\varepsilon)$ for rational $\varepsilon > 0$ is a MILP with rational data
3. a rational relaxation of X is bounded and can be perturbed into an ε-relaxation

Computable Upper Bounds

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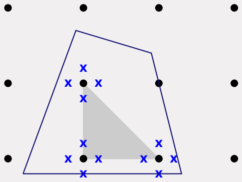

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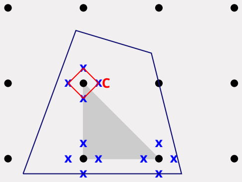

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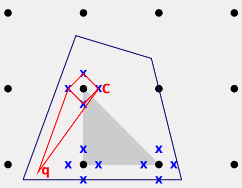
Key ideas for the proof

- variant of Minkowski's first theorem (van der Corput 1936):
  if q is too far away from C, the red region contains more than $|X|$ lattice points
- every ε-relaxation of X is contained in $\mathrm{conv}(X) + c_{d,\varepsilon,X} \cdot \mathcal{B}_d^2$, for some computable $c_{d,\varepsilon,X} > 0$



Lower Bounds

Strategy for Answering "$\mathrm{rc}(\Delta_d) = d + 1$?"

- derive a strong lower bound $\ell(X)$ on $\mathrm{rc}(X)$
- If $\ell(\Delta_d) = d + 1$, we are done.

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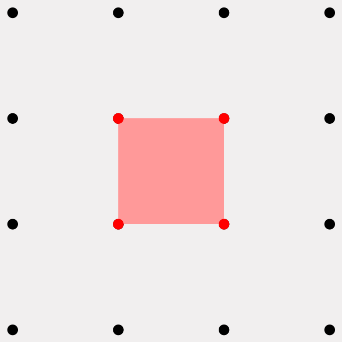
Hiding Sets and Hiding Graphs

Kaibel & Weltge 2015:

- Let $X \subseteq \mathbb{Z}^d$ be lattice-convex and $H \subseteq \text{aff}(X) \cap (\mathbb{Z}^d \setminus X)$
- H is a hiding set if, for any distinct $x, y \in H$, we have $\text{conv}(\{x, y\}) \cap \text{conv}(X) \neq \emptyset$
- For any hiding set H, $\text{rc}(X) \geq |H|$.



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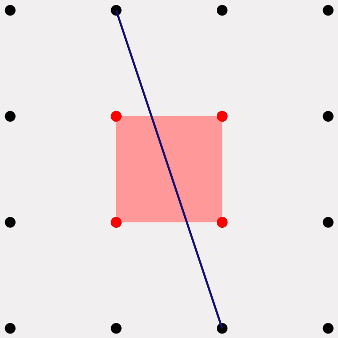

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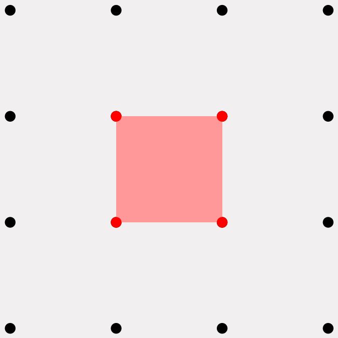
Generalization Hiding Graph:

- $G(X) = (\mathbb{Z}^d \setminus X, E)$
- $E = \{\{x, y\} \in \binom{V}{2} : x, y \text{ form hiding set}\}$
- $\mathrm{rc}(X) \geq \chi(G(X))$



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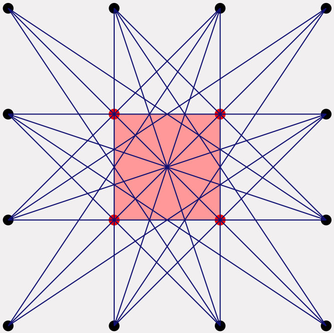

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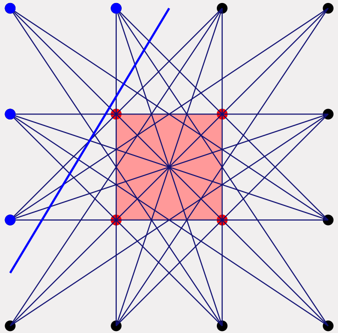

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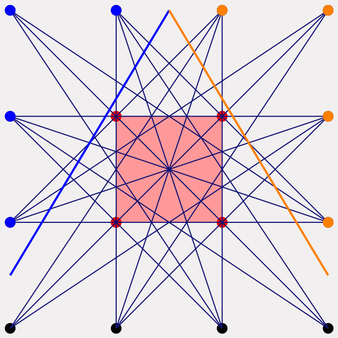

Hiding Sets and Hiding Graphs

Kaibel & Weltge 2015:

- Let $X \subseteq \mathbb{Z}^d$ be lattice-convex and $H \subseteq \mathrm{aff}(X) \cap (\mathbb{Z}^d \setminus X)$
- H is a hiding set if, for any distinct $x, y \in H$, we have $\mathrm{conv}(\{x, y\}) \cap \mathrm{conv}(X) \neq \emptyset$
- For any hiding set H, $\mathrm{rc}(X) \geq |H|$.

Generalization Hiding Graph:

- $G(X) = (\mathbb{Z}^d \setminus X, E)$
- $E = \{\{x, y\} \in \binom{V}{2} : x, y \text{ form hiding set}\}$
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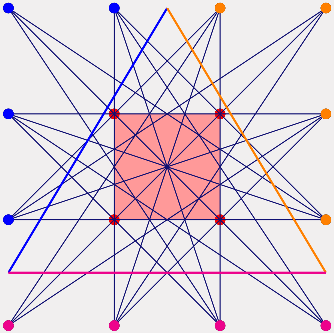

Another Lower Bound

For any $Y \subseteq \mathbb{Z}^d$, we have

$$\mathrm{rc}(X, Y) \leq \mathrm{rc}(X) \leq \mathrm{rc}_{\mathbb{Q}}(X).$$

Proposition (Averkov & Schymura 2020)

If $Y \subseteq \mathbb{Z}^d \setminus X$ is finite, $\mathrm{rc}(X, Y)$ can be computed by solving a bounded MIP.

Does there exist $Y \subseteq \mathbb{Z}^d$ finite with $\mathrm{rc}(X, Y) = \mathrm{rc}(X)$?

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Theorem (Averkov, H., & Schymura 2021)

Let $\Delta_d = \{0, e^1, \dots, e^d\}$ and $Y \subseteq \mathbb{Z}^d$ be finite. Then,

$$\mathrm{rc}(\Delta_d, Y) \leq \left\lceil \frac{d}{2} \right\rceil + 2.$$

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Consequences for Δ_d

Let $Y \subseteq \mathbb{Z}^d$ be finite.

- $\mathrm{rc}(\Delta_d, Y) \leq \lceil \frac{d}{2} \rceil + 2$

Consider restriction G' of hiding graph to nodes in Y.

- $\chi(G') \leq \lceil \frac{d}{2} \rceil + 2$

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- $\chi(G(\Delta_d)) \leq \lceil \frac{d}{2} \rceil + 2$

In particular,

- For Δ_4, all lower bounds are at most 4, whereas $\mathrm{rc}(\Delta_4) = 5$.

The Role of Rationality

The Simplex is Rebellious

For any finite $Y \subseteq \mathbb{Z}^d$,

$$\mathrm{rc}(\Delta_5, Y) \leq \mathrm{rc}(\Delta_5) \leq \mathrm{rc}_{\mathbb{Q}}(\Delta_5) = 6,$$

and

$$\mathrm{rc}(\Delta_5, Y) \leq 5.$$

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Question

Is it really true that

$$\mathrm{rc}(\Delta_5) < \mathrm{rc}_{\mathbb{Q}}(\Delta_5)?$$

Our results

Theorem (Aprile, Averkov, Di Summa, H. 2022+)

1. $5 = \mathrm{rc}(\Delta_5) < \mathrm{rc}_{\mathbb{Q}}(\Delta_5) = 6.$

Our results

Theorem (Aprile, Averkov, Di Summa, H. 2022+)

1. $5 = \mathrm{rc}(\Delta_5) < \mathrm{rc}_{\mathbb{Q}}(\Delta_5) = 6$.
2. We have $\mathrm{rc}(\Delta_d) \in O\left(\frac{d}{\sqrt{\log d}}\right)$.

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Crucial Tool—Mixed Relaxation Complexity

- let $X \subseteq Y \subseteq \mathbb{R}^k$
- polyhedron $P \subseteq \mathbb{R}^k$ is relaxation of X within Y if $X = P \cap Y$
- $\mathrm{rc}(X, Y)$: minimum number of facets of relaxation
- $Y = \mathbb{Z}^d \times \mathbb{R}^1 \rightsquigarrow$ mixed relaxation complexity

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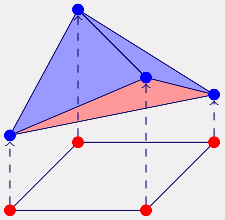
Interpretation: the continuous coordinate as height of a lifting

- $h \colon X \to \mathbb{R}$, where $X \subseteq \mathbb{Z}^d$
- $\mathrm{lift}_h(X) = \{(x, h(x)) \ : \ x \in X\}$
- $\mathrm{clift}_h(X) = \mathrm{conv}(\mathrm{lift}_h(X))$

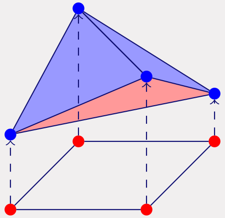


Bounding the Mixed Relaxation Complexity

Proposition

Let $P \subseteq \mathbb{R}^{k+1}$ be a full-dimensional polytope, let $T \subseteq \mathbb{Z}^k$ be finite and non-empty, and let $h\colon T \to \mathbb{R}$. Then, P is a relaxation of $\operatorname{lift}_h(T)$ within $\mathbb{Z}^k \times \mathbb{R}$ iff

- every $p \in \operatorname{lift}_h(T)$ is contained in an upper and lower facet of P, and
- the projection of P onto the first k components is a relaxation of T within $\mathbb{Z}^k$.



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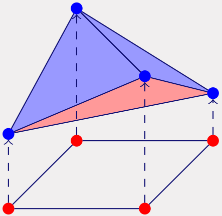

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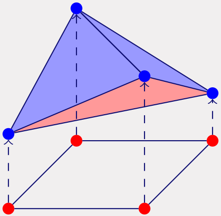

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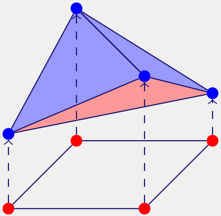

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The idea in dimension 5

- consider
$$\Delta = \{0, e^1, e^2, e^3, (1,0,1,1,0), (0,1,1,0,1)\} \subset \mathbb{Z}^5,$$
  which is unimodularly equivalent to Δ_5
- let $\ell \subset \mathbb{R}^5$ be line spanned by $(0,0,0,1,\sqrt{2})$
- Kaibel & Weltge 2015: $\mathrm{conv}(\Delta) + \ell$ is unbounded relaxation of Δ (#facets ≥ 6)

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Observation

Let $\pi : \mathbb{R}^5 \to \mathbb{R}^4$ be the projection

$$\pi(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3, x_4 - \tfrac{1}{\sqrt{2}} x_5)$$

and $P \subseteq \mathbb{R}^4$ be a polyhedron with $P \cap (\mathbb{Z}^3 \times \mathbb{R}) = \pi(\Delta)$. Then, $Q := (P \times \{0\}) + \ell$ is a relaxation of Δ.

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$$\Rightarrow \mathrm{rc}(\Delta, \mathbb{Z}^5) \leq \mathrm{rc}(\pi(\Delta), \mathbb{Z}^3 \times \mathbb{R})$$

A (3,1)-mixed relaxation

$\text{rc}(\Delta, \mathbb{Z}^5) \leq \text{rc}(\pi(\Delta), \mathbb{Z}^3 \times \mathbb{R}) \rightsquigarrow$ it is sufficient to find mixed relaxation with 5 facets

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A (3,1)-mixed relaxation of $\pi(\Delta)$ is given by the following inequalities with $\varepsilon = \frac{1}{8}$:

$$\begin{aligned} x_1 &\geq x_4 \\ x_3 &\geq x_4 \\ \varepsilon x_1 + x_2 + \frac{1-\varepsilon}{1+\sqrt{2}} x_3 + \frac{(1-\varepsilon)\sqrt{2}}{1+\sqrt{2}} x_4 &\leq 1, \\ x_3 + \sqrt{2} x_4 &\geq 0, \\ x_1 - (1+\varepsilon) x_2 + x_3 - x_4 &\leq 1. \end{aligned}$$

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Putting all pieces together

Proof of $\mathrm{rc}(\Delta_5) = 5$**:**

$$\mathrm{rc}(\Delta_5) = \mathrm{rc}(\Delta) \leq \mathrm{rc}(\pi(\Delta), \mathbb{Z}^3 \times \mathbb{R})$$

Putting all pieces together

Proof of $\mathsf{rc}(\Delta_5) = 5$**:**

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Sublinear bound on relaxation complexity

Summary of proof in dimension 5

- project (unimodular transformation of) Δ_5 onto $\mathbb{R}^4$
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$$\Delta_5: \left(\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

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The proof idea

Make calculations easier

- assume $d = 2^k - 1$.

Aim: find relaxation complexity of

$$\Delta_d = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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$$\Delta'_d = \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Step 1: projection

- project $\Delta'_{k+(d-k)}$ on the first k coordinates via π:

$$\left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

- the projection yields $\{0, 1\}^k$

Step 2: lift and estimate

Lemma

Let $Y \subseteq \mathbb{Z}^k \setminus \Delta_k$ with $|Y| = \ell$. Let $h\colon \Delta_k \cup Y \to \mathbb{R}$ be a "suitable lifting function". Then,

$$\mathrm{rc}(\Delta_{k+\ell}) \leq \mathrm{rc}(\mathrm{lift}_h(\Delta_k \cup Y), \mathbb{Z}^k \times \mathbb{R}).$$

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$$\mathrm{rc}(\Delta'_d) \leq \mathrm{rc}(\mathrm{lift}_h(\{0,1\}^k), \mathbb{Z}^k \times \mathbb{R}) \leq \mathrm{rc}(\{0,1\}^k) + \mathrm{ucn}_h(\{0,1\}^k) + \mathrm{lcn}_h(\{0,1\}^k)$$

Putting the pieces together

For a “suitable lifting” h:

$$\begin{aligned}\mathrm{rc}(\Delta_d) = \mathrm{rc}(\Delta_{k+\ell}) &\leq \mathrm{rc}(\{0,1\}^k) + \mathrm{ucn}_h(\{0,1\}^k) + \mathrm{lcn}_h(\{0,1\}^k)\\ &= (k+1) + O\Big(\frac{2^k}{\sqrt{k}}\Big)\\ &= O\Big(\frac{d}{\sqrt{\log(d)}}\Big)\end{aligned}$$

Summary

Question

Are $\mathrm{rc}(X)$ and $\mathrm{rc}_{\mathbb{Q}}(X)$, and corresponding minimal relaxations, computable?

- yes, if $d = 2$
- yes, if there exists a finite $Y \subseteq \mathbb{Z}^d$ such that $\mathrm{rc}(X, Y) = \mathrm{rc}_{\mathbb{Q}}(X)$

Question (Kaibel & Weltge 2015)

Does $\mathrm{rc}(X) = \mathrm{rc}_{\mathbb{Q}}(X)$ hold? In particular, is $\mathrm{rc}(\Delta_d) = \mathrm{rc}_{\mathbb{Q}}(\Delta_d)$ true?

- no, for every $d \geq 5$

Open questions

Computability Is $\mathrm{rc}(X)$ computable?

- Averkov & Schymura 2020: yes, if $d \leq 3$ or X has special structure

Complexity How difficult is it to compute $\mathrm{rc}(X)$ if $d = 3$?

Finite Certificate Which X admit a finite $Y \subseteq \mathbb{Z}^d \setminus X$ with $\mathrm{rc}(X, Y) = \mathrm{rc}(X)$?

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Thank you for your attention!