On Strings Having the Same Length-k Substrings

Dutch Optimization Seminar - PhD Talk

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Shortest equivalent string : problem definition

SHORTEST $\mathcal{S}$-EQUIVALENT STRING
Input: A set $\mathcal{S}$ of n length-k strings
Output: A shortest string T such that the set of k substrings of T is $\mathcal{S}$, or FAIL if that is not possible.

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The strings "abracadab" or "cadabraca" (for example) are shortest for this property

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De Bruijn graph of a string

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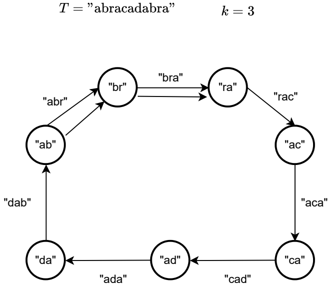

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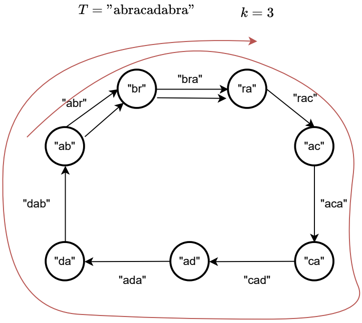

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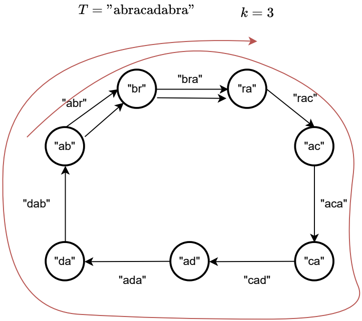
A graph G is *semi-Eulerian* if it can be traversed by visiting each edge once and only once



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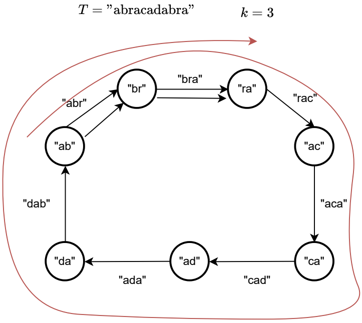
Proposition

If a graph G is a de Bruijn graph of a string, then G is semi-Eulerian



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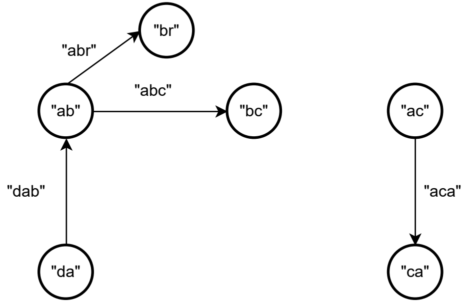
Eulerian trails

Definition

An *Eulerian trail* on a graph G is a walk on G that traverses every edges *exactly* once

De Bruijn graph of a set of k strings

$\mathcal{S}_k = \{\text{"abr"}, \text{"aca"}, \text{"dab"}, \text{"abc"}\}$ $\qquad k = 3$



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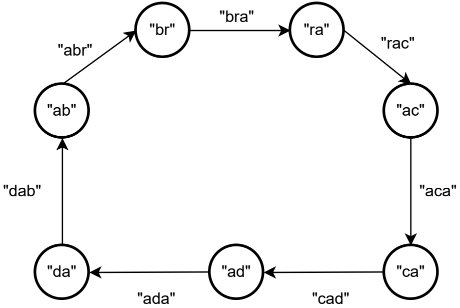

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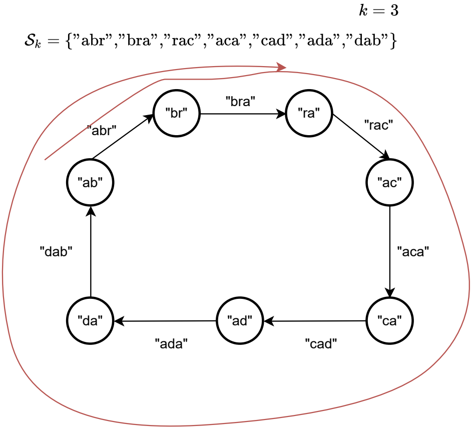

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Eulerian walks

An *Eulerian walk* on a graph G is a walk on G that traverses every edges *at least* once

Solving shortest equivalent k-string via Eulerian walks

Proposition

The strings having $\mathcal{S}$ as a set of k-substrings corresponds to walks traversing *at least* once each edge of the order k de Bruijn graph of $\mathcal{S}$.

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- We solve SHORTEST $\mathcal{S}$-EQUIVALENT STRING by :
  - Finding a minimal set of edge that can be copied to make the de Bruijn graph semi-Eulerian : we call *Eulerian extension* such a graph with copied edges

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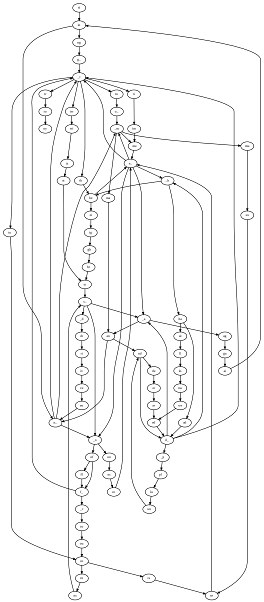
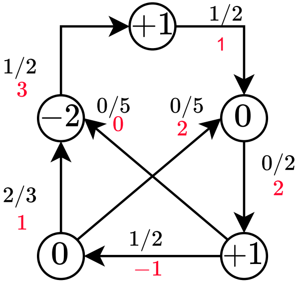
- We solve SHORTEST $\mathcal{S}$-EQUIVALENT STRING by :
  - Finding a minimal set of edge that can be copied to make the de Bruijn graph semi-Eulerian : we call *Eulerian extension* such a graph with copied edges
  - Find Eulerian trails in the graph



$T =$"sing_to_me_of_the_man_muse
_the_man_of_twists_and_turns
_driven_time_and_again_off_course
_once_he_had_plundered
_the_hallowed_heights_of_troy"
$k = 3$

Flows : definition

To find Eulerian extensions, we use flows :



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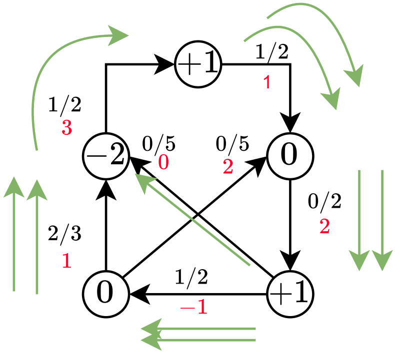

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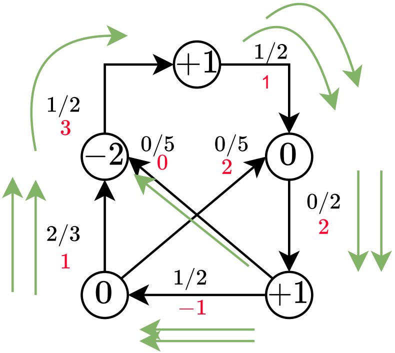
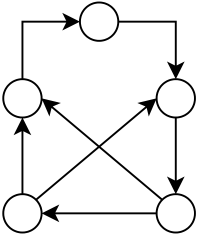


$$c(f) = 3 \cdot 1 + 1 \cdot 2 + 2 \cdot 2 - 1 \cdot 2 + 1 \cdot 2 + 0 \cdot 1 = 12$$

Construction of a flow problem



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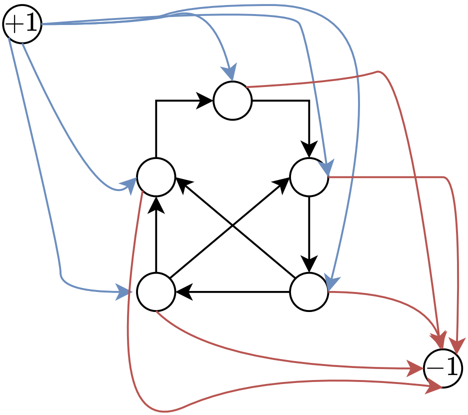

Now we want to model our extension problem with a flow problem.
For this we create bogus nodes and we connect then so the flow can "start" and "end" anywhere on the graph

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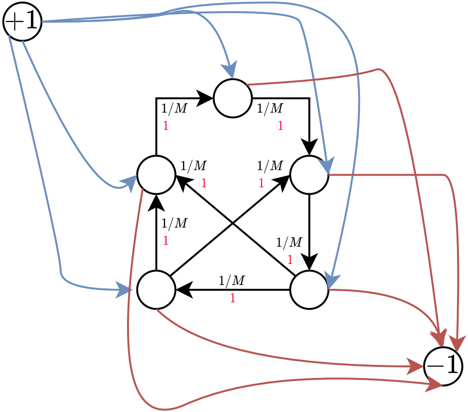

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We give minimal capacity 1 to each original edge so the flow has to visit each edge at least once

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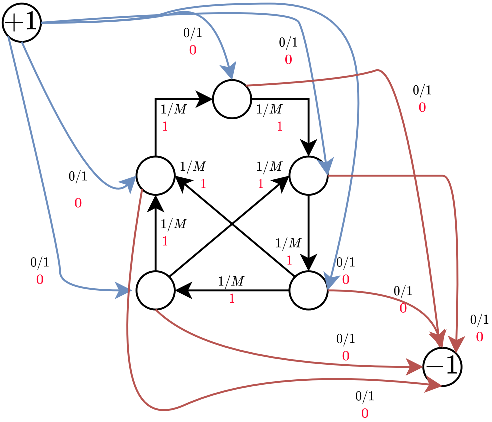

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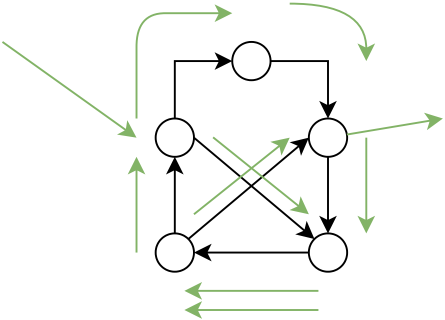
The maximal capacity is a very large integer on each edge, and the costs are 1 on edges from the original graph, 0 on bogus edges

Link between flows on G_{ext} and Eulerian extensions

- We can obtain a minimum cost flow in $\tilde{\mathcal{O}}(|E|^2 + |E||V|)$ ([Orlin, 1993])



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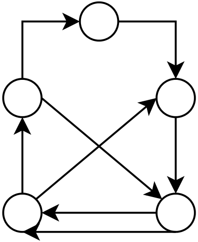

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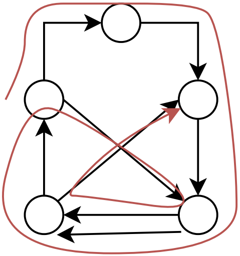

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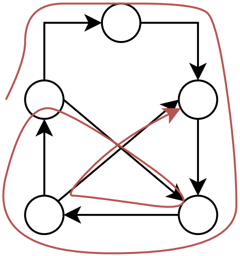

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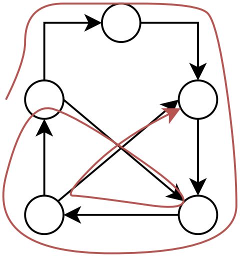
- A minimal cost flow gives us a shortest walk



Results

Theorem

The SHORTEST $\mathcal{S}$-EQUIVALENT STRING *problem can be solved in* $\tilde{\mathcal{O}}(nk + n^2)$ *time.*

Reduction : The Directed Chinese Postman

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The Directed Chinese Postman problem can be solved in $\tilde{O}(|E|^2)$ time

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The z-SHORTEST $\mathcal{S}$-EQUIVALENT STRING problem can be solved in $\tilde{\mathcal{O}}(nk + zn^2 + ||\mathcal{T}_z||)$ time.

References I

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