Sphere packing and semidedefinite programming

David de Laat (TU Delft)

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In dimension 8 and 24 the optimal configurations are the E_8 root lattice and the Leech lattice; solved using the Cohn-Elkies linear programming bound and Viazovska's (quasi)modular forms

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Each feasible solution of the dual minimization problem gives a density upper bound

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Can solve semidefinite programs efficiently using interior point methods

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Continuous linear map A between topological vector spaces E and F

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Proof: $\langle b, y \rangle = \langle Ax, y \rangle = \langle x, A^* y \rangle = \langle x, A^* y - c \rangle + \langle x, c \rangle \leq \langle x, c \rangle$ $\square$

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We say strong duality holds if $p^* = p$

Examples of pair correlation bounds

- Graph theory: Lovász ϑ-number
- Spherical codes: Delsarte-Goethals-Seidel bound
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- Conformal field theories: Modular bootstrap
- Analytic number theory: Montgomery's pair correlation approach

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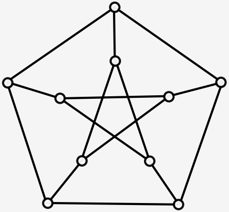

Independent set problem

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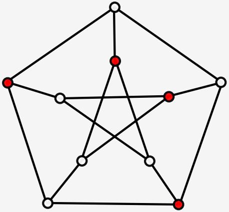

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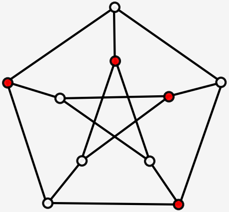

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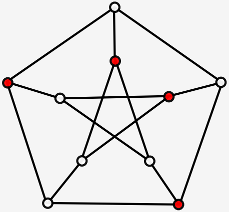

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The Lovász ϑ-number:

$$\vartheta(G) := \max\left\{\langle X, J\rangle : \langle X, I\rangle = 1,\, X_{u,v} = 0 \text{ for } u \sim v,\, X \in \mathbb{S}_+^V\right\}$$

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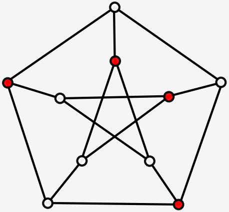

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Claim: $\alpha(G) \leq \vartheta(G)$

Proof: Let C be an independent set and define $X = \frac{1}{|C|} 1_C 1_C^{\mathsf{T}}$. $\square$

Lovász ϑ-number

The dual conic program:

$$\vartheta(G)^* = \min \Big\{ t \in \mathbb{R} : X_{u,u} = t - 1 \text{ for } u \in V, \\ X_{u,v} = -1 \text{ for distinct } x \not\sim y, \\ X \in \mathbb{S}_+^V \Big\}$$

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Strengthening by McEliece, Rodemich, and Rumsey, and Schrijver:

$$\vartheta'(G)^* = \min\Big\{t \in \mathbb{R} : X_{u,u} = t - 1 \text{ for } u \in V,\\ X_{u,v} \leq -1 \text{ for distinct } x \not\sim y,\\ X \in \mathbb{S}_+^V\Big\}$$

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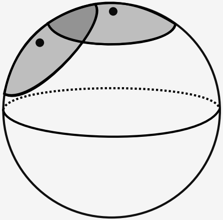

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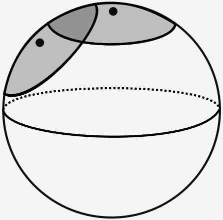

This is an independent set problem in the graph with vertex set S^{n-1} where two distinct vertices x and y are adjacent if $x \cdot y > \cos \varphi$

Delsarte-Goethals-Seidel bound

The ϑ'-number for this graph can be written as

$$\begin{aligned}\inf\{t \in \mathbb{R} : \; & K(x,x) \leq t - 1 \text{ for } x \in V, \\ & K(x,y) \leq -1 \text{ for distinct } x \not\sim y, \\ & K \colon V \times V \to \mathbb{R} \text{ continuous, positive kernel}\}\end{aligned}$$

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As observer by Bachoc, Nebe, Oliveira, and Vallentin this reduces to the Delsarte-Goethals-Seidel bound

$$\begin{aligned}\inf\{f(1) : \; & f(t) = \sum_{n=0}^{\infty} f_n Q_n(t), \\ & f(t) \le 0 \text{ for } t \in [-1, \cos\theta] \\ & f_0 = 1, f_1, f_2, \ldots \ge 0, \sum_n f_n < \infty\},\end{aligned}$$

where the $Q_0, Q_1, \ldots$ are the ultraspherical polynomials for S^{n-1}

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How do we find an exact optimal solution?

Exact bounds from semidefinite programming

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Implementation supports rounding to rationals and quadratic fields

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Proof for lattice sphere packings: If f is feasible and Λ is a lattice with minimal vector length 1, then

$$f(0) \geq \sum_{x \in \Lambda} f(x) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda^*} \widehat{f}(x) \geq \frac{1}{|\Lambda|} \widehat{f}(1) = \frac{1}{|\Lambda|}. \quad \square$$

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In 2016 Viazovska defines optimal function f in 8 dimensions by the analytic continuation of the function

$$\sin^2\left(\pi\|x\|^2\right)\int_0^\infty (g_+(t) + g_-(t))e^{-\pi\|x\|^2 t}\,dt,$$

defined on $(1, \infty)$, where $g_\pm$ are (quasi)modular forms such that

$$\widehat{f}(x) = \sin^2\left(\pi\|x\|^2\right)\int_0^\infty (g_+(t) - g_-(t))e^{-\pi\|x\|^2 t}\,dt$$

Define g_+ and g_- such that $g_+ + g_- < 0$ and $g_+ - g_- > 0$ on $(0, \infty)$

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Up to subexponential factors this is still the best known asymptotic lower bound

Asymptotic sphere packing upper bounds

The Kabatianskii-Levenshtein bound:

$$\Delta_{\mathbb{R}^n} \leq 2^{-(0.59905576+o(1))n}$$

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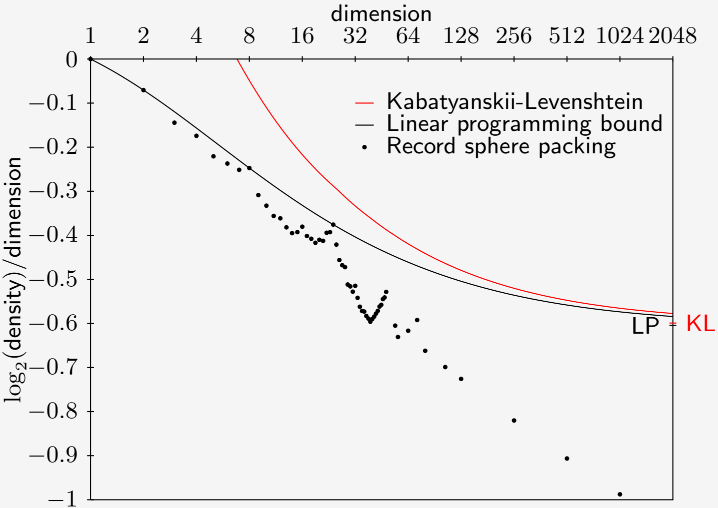
In 2014 Cohn and Zhao show we can interpret this bound in terms of feasible solutions to the Cohn-Elkies bound

Asymptotic sphere packing upper bounds



N. Afkhami-Jeddi, H. Cohn, T. Hartman, D. de Laat and A. Tajdini, High-dimensional sphere packing and the modular bootstrap, Journal of High Energy Physics (2020)

Asymptotic sphere packing upper bounds

Conjecture

The Cohn-Elkies bound proves

$$\Delta_{\mathbb{R}^n} \leq 2^{-(\lambda+o(1))d}$$

for some $0.604 < \lambda < 0.605$ when the auxiliary function is fully optimized

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The constant λ is given by $2^{-\lambda} = \sqrt{e/(2\pi)}$.

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Pair correlation bounds

- Graph theory: Lovász ϑ-number
- Spherical codes: Delsarte-Goethals-Seidel bound
- Sphere packing: Cohn-Elkies linear programming bound
- **Conformal field theories: Modular bootstrap**
- Analytic number theory: Montgomery's pair correlation approach

A modular bootstrap problem

Partition functions of certain conformal field theories can be written as

$$\mathcal{Z}(\tau) = \chi_0(\tau) + \sum_{\Delta > 0} d_\Delta \chi_\Delta(\tau),$$

where

$$\chi_\Delta(\tau) = \frac{e^{2\pi i \tau \Delta}}{\eta(\tau)^{2c}} \quad \text{and} \quad \mathcal{Z}(-1/\tau) = \mathcal{Z}(\tau)$$

and where η is the Dedekind eta function

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How large can the the spectral gap be?

The parameter here is c and the variables are

$$0 < \Delta_1 < \Delta_2 < \dots \quad \text{and} \quad d_{\Delta_k} \in \mathbb{N}_1 \text{ for } k \geq 1$$

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In 2019 Hartman, Mazáč, and Rastelli show optimizing over ω is exactly the Cohn-Elkies bound in dimension $2c$ by formulating it as an uncertainty problem (up to a widely believed conjecture)

Examples of pair correlation bounds

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Simple zeros of the zeta function

The Riemann zeta function is the analytic continuation to $\mathbb{C} \setminus \{1\}$ of

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Goal: Find small $c \geq 1$ for which we can prove (under RH or GRH):

$$N^*(T) \leq (c + o(1))N(T)$$

Simple zeros of the zeta function

Under RH we have

$$N^*(T) \leq (c + o(1))N(T),$$

with

$$c = \inf \left\{ f(0)+2\int_0^1 f(x)x\,dx : \right.$$
$$\left. f \in S(\mathbb{R}),\ \widehat{f}(0) = 1,\ \widehat{f} \geq 0,\ f(x) \leq 0 \text{ for } |x| \geq 1 \right\},$$

A. Chirre and F. Gonçalves, D. de Laat, Pair correlation estimates for the zeros of the zeta-function via semidefinite programming, Advances in Mathematics (2020)

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This is a small improvement on existing techniques by Montgomery, Cheer, and Goldston. Comparable to replacing ϑ by ϑ'!

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Thank you!