A Faster Exponential Time Algorithm for Bin Packing with Constant Number of Bins

Jesper Nederlof

UU

Jakub Pawlewicz

Warsaw

Céline Swennenhuis

TU/e

Karol Węgrzycki

Saarbrucken

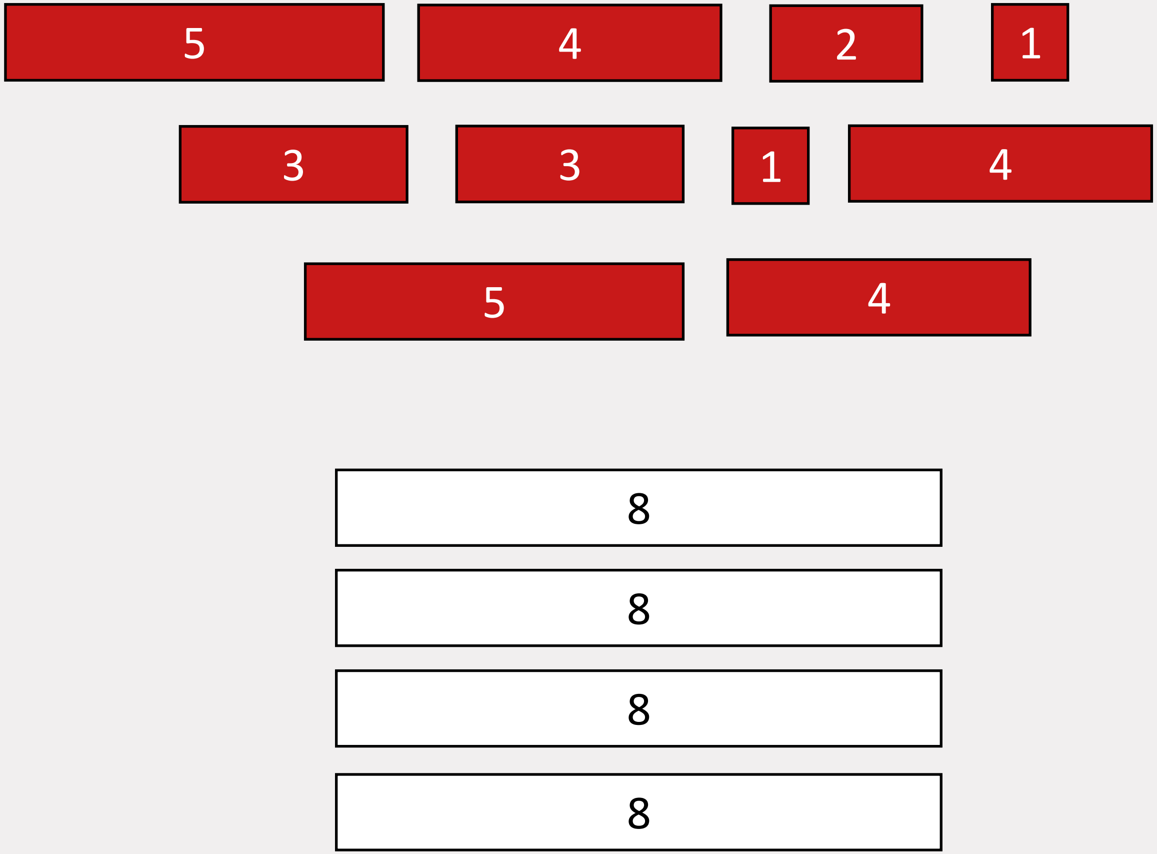
Department of Mathematics and Computer Science, Eindhoven University of Technology

Bin Packing

Given:

- n items
- $w(j)$ weight of item j
  - ➢ $w(X) = \sum_{j \in X} w(j)$
- m bins with capacity c

Goal: distribute items over bins



Bin Packing

- Algorithm A
  - Dynamic Programming
  - $O(c^m \cdot n)$
- Algorithm B
  - Björklund, Husfeldt and Koivisto (SICOMP 2009)
  - $O(2^n \cdot n)$

Open Question

Can do in $O(1.99999^n)$ time?

Before our work, only known for $m = 2,3$ (Lente et al.)

Our result:

Can do in $O\left((2-\varepsilon_m)^n\right)$ time with $\varepsilon_m > 0$ that depends on m.

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Bin Packing

$$\alpha(w) = |\{w(X): X \subseteq [n]\}|$$
$$\mathcal{B}(w) = \{X: w(X) = s\}$$

- Algorithm A
  - Dynamic Programming
  - $O(c^m \cdot n)$
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Key idea:

Algorithm A in $O\left((1.999)^n\right)$ time

If $\alpha(w)$ is small

or

Algorithm B in $O\left((1.999)^n\right)$ time

If $\mathcal{B}(w)$ is small

Open Question

Can do in $O(1.99999^n)$ time?

Before our work, only known for $m = 2{,}3$ (Lente et al.)

Our result:

Can do in $O\left((2 - \varepsilon_m)^n\right)$ time with $\varepsilon_m > 0$ that depends on m.

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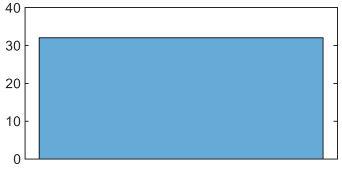
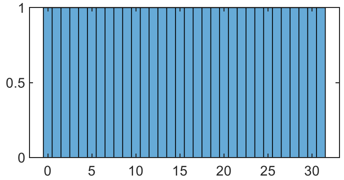
Parameters $\alpha(w)$ and $|\mathcal{B}|$

$$\alpha(w) = |\{w(X) : X \subseteq [n]\}|$$
$$\mathcal{B}(w) = \{X : w(X) = s\}$$

$w_1, \dots, w_5$	Histogram	$\alpha(w)$	$\max|\mathcal{B}|$
0 0 0 0 0		1	32
1 2 4 8 16		32	1

Algorithm A:

If $\alpha(w) \leq 1.99^{n/m}$
Algorithm A runs in time
$O(1.99^n n^2) = O(1.999^n)$

$$T(j, c_1, c_2, \ldots, c_m) = \bigvee_{i \in [m]} T(j-1, c_1, \ldots, c_i - w(j), \ldots, c_m)$$

Items $1, \ldots, j$ — leftover capacities

Runtime: $O(c^m \cdot n^2)$

$$\boldsymbol{\alpha(w) = |\{w(X): X \subseteq [n]\}|}$$

New runtime: $O(\alpha(w)^m \cdot n^2)$

Algorithm B:

Theorem: For $\boldsymbol{\mathcal{B}} \subseteq \mathbf{2^n}$, we can compute in time $O(|{\downarrow} \mathcal{B}| \cdot n)$, whether X can be distributed over m bins **for all $\boldsymbol{X} \in \boldsymbol{\mathcal{B}}$**.

${\downarrow} \mathcal{B}$ = all **subsets** of a set in $\mathcal{B}$

$\mathcal{B} = \{X_1, X_2, X_3, \ldots\}$

Yes!

No!

Yes!

Based on:

- Inclusion/Exclusion
- Fast algebraic transformations (Zeta/Möbius)

Algorithm B:

Theorem: For $\mathcal{B} \subseteq 2^n$, we can compute in time $O(|\downarrow \mathcal{B}| \cdot n)$, whether X can be distributed over m bins **for all $X \in \mathcal{B}$**.

$\downarrow \mathcal{B}$ = all **subsets** of a set in $\mathcal{B}$

Assume:
$\frac{n}{2}$ items fit **exactly** in $\frac{m}{2}$ bins

$$\mathcal{B} := \left\{X : |X| = \frac{n}{2}, w(X) = c \cdot m/2\right\}$$

Compute for each $X \in \mathcal{B}$ if X fits in $m/2$ bins

$O(|\downarrow \mathcal{B}| \cdot n)$

$$|\mathcal{B}| \leq \left|\left\{X : w(X) = c \cdot \frac{m}{2}\right\}\right| \leq |\mathcal{B}(w)|$$

$m/2$	$m/2$
X	$[n] \setminus X$
Yes!	Yes!

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Algorithm B:

If $|\mathcal{B}(w)| \leq 1.99^n$,
Algorithm B runs in time $O(1.999^n)$

Theorem: For $\boldsymbol{\mathcal{B} \subseteq 2^n}$, we can compute in time $O(|\downarrow \boldsymbol{\mathcal{B}}| \cdot n)$, whether X can be distributed over m bins **for all $\boldsymbol{X \in \mathcal{B}}$**.

$\downarrow \mathcal{B}$ = all **subsets** of a set in $\mathcal{B}$

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Compute for each $X \in \mathcal{B}$ if X fits in $m/2$ bins

$O(|\downarrow \mathcal{B}| \cdot n)$

$$|\mathcal{B}| \leq \left|\left\{X : w(X) = c \cdot \frac{m}{2}\right\}\right| \leq |\mathcal{B}(w)|$$

$$|\mathcal{B}| \leq 1.99^n \Rightarrow |\downarrow \mathcal{B}| \leq 1.999^n$$

$\alpha(w) = |\{w(X): X \subseteq [n]\}|$
$\mathcal{B}(w) = \{X: w(X) = s\}$

Additive Combinatorics

If $\alpha(w) \leq 1.99^{n/m}$
Algorithm A runs in time
$O(1.999^n)$

If $|\mathcal{B}(w)| \leq 1.99^n$,
Algorithm B runs in time
$O(1.999^n)$

Additive Combinatorics

$\alpha(w) = |\{w(X): X \subseteq [n]\}|$
$\mathcal{B}(w) = \{X: w(X) = s\}$

If $\alpha(w) \leq 1.99^{n/m}$
Algorithm A runs in time
$O(1.999^n)$

If $|\mathcal{B}(w)| \leq 1.99^n$,
Algorithm B runs in time
$O(1.999^n)$

Theorem: For all $\delta > 0$ there exists $\varepsilon > 0$ s.t.

if $\alpha(w) \geq 2^{\delta n}$, then $|\mathcal{B}(w)| \leq 2^{(1-\varepsilon)n}$.

How to prove this?

Take $\delta < \frac{1}{m}$, then either

$\alpha(w) \leq 2^{\delta n} \leq 1.99^{n/m}$ or $|\mathcal{B}(w)| \leq 2^{(1-\varepsilon)n} \leq 1.99^n$ *(or both)*

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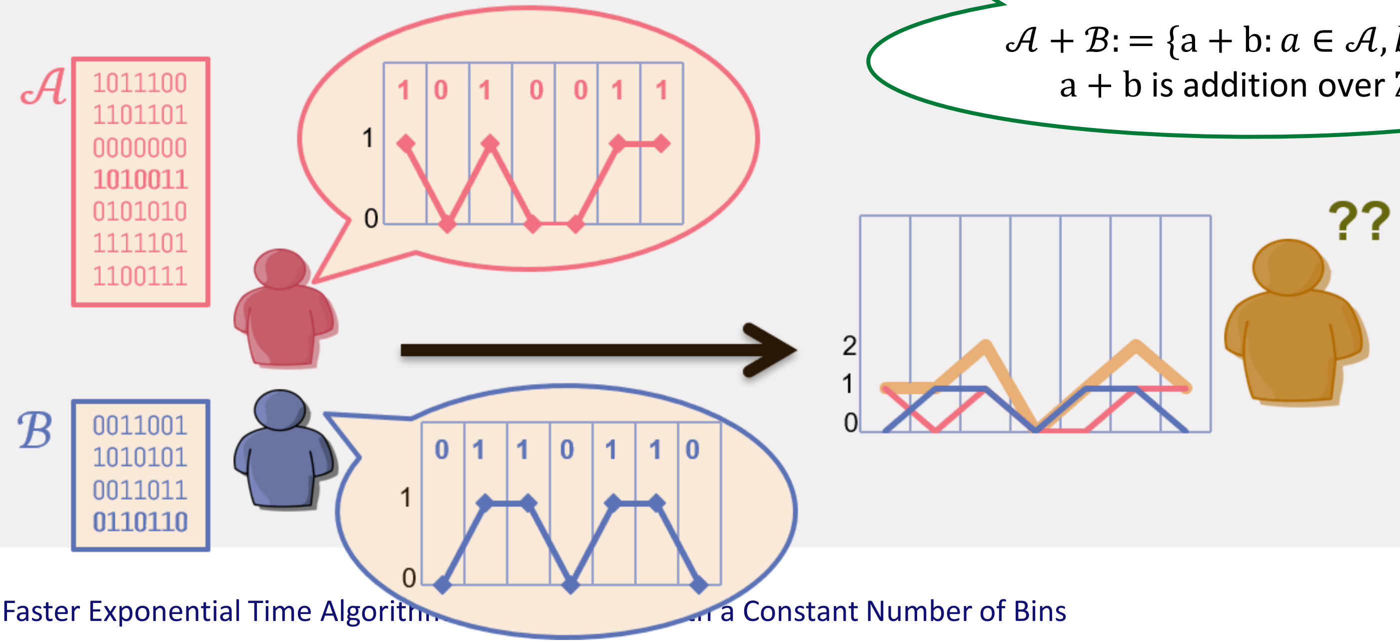
Additive Combinatorics

Def: Uniquely Decodable Pairs (UDCP's)

A UDCP is a pair $\mathcal{A}, \mathcal{B} \subseteq \{0,1\}^n$ s.t. $|\mathcal{A}+\mathcal{B}| = |\mathcal{A}||\mathcal{B}|$.

$\mathcal{A}+\mathcal{B} := \{\mathrm{a}+\mathrm{b} : a \in \mathcal{A}, b \in \mathcal{B}\}$
a + b is addition over $\mathbb{Z}^n$.



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a + b is addition over $\mathbb{Z}^n$.

Example 1:

$\mathcal{A} = \{000, 100\}$ $\mathcal{B} = \{000, 001, 010, 011\}$

$\mathcal{A} + \mathcal{B} = \{000, 001, 010, \ldots\}$

Example 2:

$\mathcal{A} = \{10, 01\}$ $\mathcal{B} = \{00, 01, 11\}$

$\mathcal{A} + \mathcal{B} = \{10, 11, 21, 01, 02, 12\}$

$\alpha(w) = |\{w(X) : X \subseteq [n]\}|$
$\mathcal{B}(w) = \{X : w(X) = s\}$

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$\mathcal{A} \subseteq \{0,1\}^n$ s.t. all $a \in \mathcal{A}$ **have different weight** $|\mathcal{A}| = \alpha(w)$
$\mathcal{B} \subseteq \{0,1\}^n$ s.t. all $b \in \mathcal{B}$ **have weight s** $|\mathcal{B}| = |\mathcal{B}(w)|$

$\mathcal{A}$ and $\mathcal{B}$ is UDCP:
Let c be received.
$c = a + b$, so $\langle w, c \rangle = \langle w, a \rangle + \langle w, b \rangle$.
$\Rightarrow a$ was used!
$b = c - a$.

See w as a vector,

$$w(X) = \langle w, X \rangle = \left\langle \begin{matrix} w(1) \\ w(2) \\ w(3) \\ \vdots \\ w(n) \end{matrix}, \begin{matrix} 0 \\ 1 \\ 0 \\ \vdots \\ 1 \end{matrix} \right\rangle$$

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Additive Combinatorics

Def: Uniquely Decodable Pairs (UDCP's)

A UDCP is a pair $\mathcal{A}, \mathcal{B} \subseteq \{0,1\}^n$ s.t. $|\mathcal{A} + \mathcal{B}| = |\mathcal{A}||\mathcal{B}|$.

Best known bounds: *If $\mathcal{A}, \mathcal{B}$ is UDCP:*

$|\mathcal{A}| \cdot |\mathcal{B}| \leq 2^{1.5\,n}$, [Tilborg, 1978]

If $|\mathcal{A}| \geq 2^{(1-\varepsilon)n}$ then $|\mathcal{B}| \leq 2^{(0.4228+\sqrt{\varepsilon})n}$. [Austrin et al. 2018]

We need: If $|\mathcal{A}| \geq 2^{\delta n}$, then $|\mathcal{B}| \leq 2^{(1-\varepsilon)n}$.

$\alpha(w) = |\{w(X): X \subseteq [n]\}|$
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Additive Combinatorics

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$\mathcal{A}$ and $\mathcal{B}$ is UDCP.

$k \cdot \mathcal{B} = \{b_1 + \cdots + b_k : b_i \in \mathcal{B}\}$

$\langle w, b\rangle = k \cdot s$ for all $b \in k \cdot \mathcal{B}$.

$\mathcal{A}$ and $k \cdot \mathcal{B}$ is \`UDCP'!

Assume $\mathcal{B} \approx \{0,1\}^n$

$\subseteq \{0, \ldots, k+1\}^n$

$$|\mathcal{A}| \leq \frac{|\mathcal{A} + k \cdot \mathcal{B}|}{|k \cdot \mathcal{B}|} \leq \frac{(k+2)^n}{(k+1)^n} = \left(1 + \frac{1}{k+1}\right)^n = 2^{\delta_k n}$$

$\approx \{0, \ldots, k\}^n$

$\delta_k \to 0$
as $k \to \infty$

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Conclusion

Main result:

Bin Packing in $O\left((2-\varepsilon_m)^n\right)$ time with $\varepsilon_m > 0$ that depends on m.

Key idea:

Tradeoff between $\alpha(w)$ and $\mathcal{B}(w)$.

Future Research:

Bin Packing in $O(1.9999^n)$, m not a constant!

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