Modelling and Computing the Total Value Adjustment for European Derivatives in a Multi-Currency Setting

Iñigo Arregui, Roberta Simonella, and Carlos Vázquez

Abstract Since the global financial crisis of 2007–2008, different adjustments are considered in the pricing of financial products to incorporate the counterparty risk; the set of these adjustments is referred to as total value adjustment or XVA. In this work we first pose a partial differential equations (PDE) model for pricing the XVA associated to European-like derivatives in multi-currency situations. Moreover, we formulate and solve the XVA pricing problem in terms of expectations to overcome the curse of dimensionality arising in PDEs formulation. Numerical results illustrate the performance of the proposed Monte Carlo algorithms to price best-of-all call options and the sum of put options denominated in different currencies. The second example additionally illustrates the appropriate scaling when the number of stochastic factors (currencies) becomes large.

1 Statement of Partial Differential Equations Model

As a consequence of the financial crisis of 2007–2008, it was clear that the possibility of counterparties default should be taken into account in the pricing of financial derivatives by means of appropriate valuation adjustments, either related to credit (CVA), funding (FVA) or collateral (CollVA), for example. More recently, adjustements related to capital (KVA) or margin (MVA) have been considered. We address the reader to the books [5, 9, 10] and the references therein. In the single currency framework three main approaches have been developed. A first one based on PDEs with seminal references [6, 16], the second one based on expectations started with [4], and the third one based on backward stochastic differential equations [7, 8].

In the present work we consider a multi-currency setting, following the ideas in [12], where the joint consideration of CVA, FVA, CollVA and repo adjustments

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are taken into account. We will refer to the set of this adjustments as total value adjustment or XVA. For the additional inclusion of KVA or MVA in the XVA, the ideas in [14, 13] in the single currency case could be considered.

In this section we pose a PDE formulation for the value of a derivative traded in a multi-currency framework, taking into account the total value adjustment to consider possible defaults of the counterparties involved in the deal.

Let $S_t = (S_t^1, \ldots, S_t^N)$ be the vector, at time t, of the underlying assets prices S_t^i, $i = 1, \ldots, N$, each one of them being denominated in its corresponding *foreign* currency C_i. Moreover, let h_t be the investor's credit spread, and X_t^{D,C_j} (for $j = 0, \ldots, N$) the foreign exchange (FX) rate between the *domestic* currency D and C_j, namely the domestic price of one unit of the foreign currency C_j.

The stochastic differential equations (SDEs) governing the evolution of the prices of the underlying assets, the FX rates (see [5]), and the investor's credit spread under the risk neutral probability measure (Q^D) of the domestic market are:

$$dS_t^i = (r^i - q^i) S_t^i \, dt + \sigma^{S^i} S_t^i \, dW_t^{S^i}, \qquad i = 1, \ldots, N, \tag{1}$$

$$dX_t^{D,C_j} = (r^D - r^j) X_t^{D,C_j} \, dt + \sigma^{X^j} X_t^{D,C_j} \, dW^{X^j}, \quad j = 0, \ldots, N, \tag{2}$$

$$dh_t = -\kappa \frac{h_t}{1-R} \, dt + \sigma^h \, dW_t^h, \tag{3}$$

where r^D and r^i are respectively the risk-free rate in currencies D and C_i, q^i is the dividend paid by S^i, and R is the investor's recovery rate. Moreover, σ^{S^i}, σ^{X^j} and σ^h are the volatility functions of S_t^i, X_t^{D,C_j} and h_t, respectively, while W^{S^i}, W^{X^j} and W^h are correlated Brownian motions. Nevertheless, in the following we consider $\sigma^{X^j} = 0$ in order to have deterministic FX rates.

Next, let J_t^P be the investor's default state at time t, i.e., $J_t^P = 1$ in case of default before or at time t, otherwise $J_t^P = 0$. We use the notation $V_t^D = V^D(t, S_t, h_t, J_t^P)$ for the derivative value at time t from the investor's point of view in domestic currency and $V_t^{RF,D} = V^{RF,D}(t, S_t)$ for the corresponding risk-free derivative price, i.e, traded between two non-defaultable counterparties.

In order to price the derivative, we follow [11, 12] and consider a self-financing portfolio Π that hedges all the risk factors: the market risk due to changes in $S^1, S^2, \ldots, S^N$, the investor's spread risk due to changes in h, and the investor's default risk. Moreover, we assume the existence of a collateral account, denominated in currency C_0, composed of a portfolio of bonds R^{C_0} and cash M^{C_0}. We address the reader to [2] for further details.

No arbitrage arguments and the self-financing condition, jointly with the use of Itô's formula for jump-diffusion processes, lead to the following pricing PDE for a European-like derivative with counterparty risk (see [2], for details):

$$\begin{aligned}\frac{\partial V^D}{\partial t} &+ \mathcal{L}_{Sh} V^D - f^{H,D} V^D + \frac{h}{1-R} \Delta V^D \\ &= \left[(r^R + b^{D,C_0} - f^{H,D}) R^{C_0} + (c^D + b^{D,C_0} - f^{H,D}) M^{C_0} \right] X^{D,C_0}, \end{aligned} \tag{4}$$

where $\mathcal{L}_{Sh}$ is the second order differential operator given by

$$\begin{aligned}\mathcal{L}_{Sh} = &\frac{1}{2}\sum_{i,k=1}^{N}\rho^{S^iS^k}\sigma^{S^i}\sigma^{S^k}S^iS^k\frac{\partial^2}{\partial S^i\partial S^k}+\frac{1}{2}(\sigma^h)^2\frac{\partial^2}{\partial h^2}\\ &+\sum_{i=1}^{N}\rho^{S^ih}\sigma^{S^i}\sigma^hS^i\frac{\partial^2}{\partial S^i\partial h}+\sum_{i=1}^{N}(r^i-q^i)S^i\frac{\partial}{\partial S^i}-\frac{\kappa h}{1-R}\frac{\partial}{\partial h},\end{aligned} \tag{5}$$

and ΔV^D is the variation of V^D upon default defined as $\Delta V^D = RM^+ + M^- - V^D$, with $M(t, S_t, h_t)$ representing the mark-to-market derivative price.

Two possible values for M are usually chosen [6]: either equal to the risky value or to the risk-free value of the derivative. We choose $M = V^D$, so that (4) turns into

$$\frac{\partial V^D}{\partial t} + \mathcal{L}_{Sh}V^D - fV^D = (\bar{r}R^{C_0} + \bar{m}M^{C_0})X^{D,C_0} + h(V^D)^+, \tag{6}$$

where $\bar{r} = r^R + b^{D,C_0} - f^{H,D}$, $\bar{m} = c^D + b^{D,C_0} - f^{H,D}$ and $f = f^{H,D}$.

Next, we denote by U the XVA price, that can be computed as the difference between the risky derivative value V^D and the risk-free derivative value $V^{RF,D}$. As V^D and $V^{RF,D}$ are both equal to the payoff at maturity T, we have $U(T, S, h) = 0$.

Considering that the risk-free price follows the multidimensional Black-Scholes equation, from (6) we obtain the following nonlinear PDE for the XVA price [2]:

$$\frac{\partial U}{\partial t} + \mathcal{L}_{Sh}U - fU = h\big(V^{RF,D} + U\big)^+ + \big(\bar{r}R^{C_0} + \bar{m}M^{C_0}\big)X^{D,C_0}, \tag{7}$$

jointly with the final condition $U(T, S, h) = 0$, where $(t, S, h) \in [0, T) \times (0, +\infty)^N \times (0, +\infty)$. As an alternative, the choice $M = V^{RF,D}$ leads to a linear model [2].

2 Formulation in Terms of Expectations

Since the spatial dimension of (7) increases with the number of currencies, the PDE easily becomes high dimensional in space. Therefore, we propose in this section an alternative expectation-based formulation. In this way, we overcome the so-called *curse of dimensionality*, that affects most of the numerical approaches to solve PDE problems. Thus, we use a Monte Carlo method to approximate expectations in a multidimensional framework, allowing to manage problems that involve more than two stochastic factors.

In order to compute the values of U by using the Monte Carlo method, we apply the nonlinear Feynman-Kac theorem [15], that relates the solution of nonlinear PDEs with the solution of BSDEs. More precisely, Theorem 1.1 in [3] can be applied to formulate (7) in terms of the following nonlinear integral equation:

$$U(t,S,h) = E_t^{Q^D}\Bigg[-\int_t^T e^{-f(u-t)}\Big(h_u\Big(V^{RF,D}(u,S_u)+U(u,S_u,h_u)\Big)^+ + \Big(\bar{r}R_u^{C_0}+\bar{m}M_u^{C_0}\Big)X_u^{D,C_0}\Big)du \mid S_t = S, h_t = h\Bigg]. \quad (8)$$

Analogously to [12], the integrand in the first line of (8) corresponds to CVA+FVA, while the integrand in the second line is related to CollVA and repo adjustment.

In order to compute the XVA given at time $t = 0$, i.e. when the derivative is priced, we numerically solve (8) with a fixed point method and a trapezoidal quadrature formula. Thus, we start from $U^0 = 0$ and recursively compute until convergence:

$$U^{\ell+1}(0,S,h) = E_0^{Q^D}\Bigg[-\int_0^T e^{-fu}\Big(h_u\Big(V^{RF,D}(u,S_u)+U^{\ell}(u,S_u,h_u)\Big)^+ + \Big(\bar{r}R_u^{C_0}+\bar{m}M_u^{C_0}\Big)X_u^{D,C_0}\Big)du \,\Bigg|\, S_0 = S, h_0 = h\Bigg].$$

3 Numerical Results

We now report some results obtained by using the Monte Carlo method for the evaluation of different multi-asset options in the presence of XVA. In all the examples we have considered constant FX rates and maturity T has been set to 6 months. The values of the parameters are specified in Table 1. Moreover, we have used $N_P = 10000$ paths and $N_T = 1000$ time steps. Other test cases are presented in [2].

Table 1: Financial data.

$r^1 = 0.30$	$r^2 = 0.24$	$h_0 = 0.20$	$\rho^{S^1S^2} = 0.15$	$R_0^D = 15$	$f = 0.06$
$q^1 = 0.24$	$q^2 = 0.18$	$R_C = 0.30$	$\rho^{S^1h} = 0.40$	$M_0^D = 15$	$\bar{r} = 0.01$
$\sigma^{S^1} = 0.30$	$\sigma^{S^2} = 0.20$	$\kappa = 0.01$	$\rho^{S^2h} = -0.20$		$\bar{m} = 0.02$

In the first example we assume the default-free hedger H buys, from a defaultable counterparty C, a European best-of-all call option, the payoff of which is given by:

$$G(t,S^1,S^2) = \max\big((X^{D,C_1}S^1-K^1)^+,(X^{D,C_2}S^2-K^2)^+\big), \quad (9)$$

where S^1 and S^2 are two assets respectively written in currencies C_1 and C_2, and K^1, K^2 are the strike values given in the domestic currency D. In our numerical tests, we have chosen $K^1 = 12$ and $K^2 = 15$.

Figure 1 shows the risky option price and XVA, the latter being negative because H asks the counterparty C for a reduction in the price since C may default. Table 2,

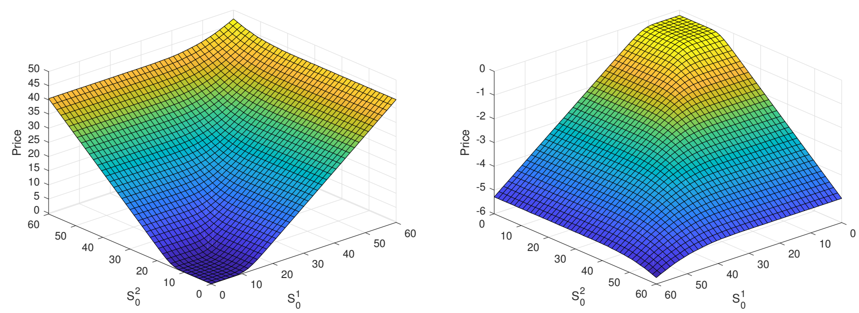

Fig. 1: Best-of-all call option. Price of the risky option (left) and total value adjustment (right).

where the notation $S^{i,D} = X^{D,C_i} S^i$ has been used, shows Monte Carlo 99 % confidence intervals for option prices and XVA values for different initial asset prices and investor's credit spread values. Since the credit spread represents the probability of C's default, the XVA value becomes more negative when increasing h.

Table 2: Best-of-all call option. Monte Carlo confidence intervals.

		$S^{1,D} = 10$		$S^{1,D} = 14$	
$S^{2,D}$	h	V^D	XVA	V^D	XVA
12	0.10	[0.1200,0.1681]	[-0.2401,-0.2394]	[2.2464,2.3812]	[-0.3591,-0.3542]
12	0.15	[0.1128,0.1609]	[-0.2473,-0.2466]	[2.1785,2.3133]	[-0.4270,-0.4220]
12	0.20	[0.1055,0.1535]	[-0.2547,-0.2539]	[2.1089,2.2437]	[-0.4967,-0.4916]
18	0.10	[3.1003,3.2270]	[-0.4057,-0.3991]	[3.9446,4.0757]	[-0.4521,-0.4439]
18	0.15	[3.0087,3.1354]	[-0.4974,-0.4907]	[3.8293,3.9603]	[-0.5674,-0.5591]
18	0.20	[2.9146,3.0413]	[-0.5915,-0.5847]	[3.7109,3.8420]	[-0.6859,-0.6774]

In the second example we consider that the non-defaultable hedger H buys, from a defaultable counterparty C, a portfolio of N European put options denominated in different currencies, so that the portfolio payoff function is given by:

$$G(t, S^1, \ldots, S^N) = \sum_{i=1}^{N} (K^i - X^{D,C_i} S^i)^+ \,, \tag{10}$$

where S^i ($i = 1, \ldots, N$) are the prices of the underlying assets, respectively written in currencies C_i, while K^i are the respective strike values in the domestic currency

D for each put option. Table 3 shows the Monte Carlo 99 % confidence intervals for the risk-free, risky and XVA prices for different numbers of underlying assets. Moreover, the elapsed computational time is reported, thus showing a linear increase with the number of assets (stochastic factors). The initial assets prices and the strike values lie in the interval $[10, 18]$.

Table 3: Sum of put options. Monte Carlo confidence intervals and elapsed time.

Assets	$V^{RF,D}$	V^D	XVA	Time (s)
2	[4.9927, 5.1547]	[4.2446, 4.4104]	[-0.7510,-0.7414]	1.0167
8	[18.7710,19.1090]	[16.5670,16.9110]	[-2.2186,-2.1827]	3.3562
32	[65.9540,66.4760]	[58.7910,59.3200]	[-7.2220,-7.0965]	16.530

Finally, we restrict our analysis to the case of the sum of two put options and we set $K^1 = 20$ and $K^2 = 25$. Note that the XVA is negative because the buyer of the derivative H will ask the counterparty C for a reduction in the price due to the potential default of C. As shown in Figure 2, the XVA becomes more negative when the option is in the money, namely when the asset prices are lower, because H would be more affected by C's default, while the XVA approaches to zero if the asset prices increase, so that the option becomes out of the money. Moreover, the XVA becomes more negative when increasing the number of assets which increases the payoff so that H is more affected by C's default.

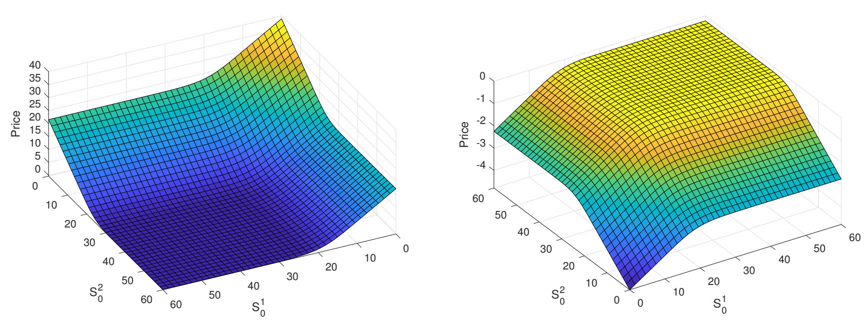

Fig. 2: Sum of put options. Price of the risky option (left) and total value adjustment (right).

4 Conclusions

With the aim of modelling the total value adjustment in a multi-currency setting, we have extended our methodology [1]. Thus, we have stated a nonlinear model and proposed a Monte Carlo method to compute the XVA, that overcomes the curse of dimensionality. We show the suitable performance of the proposed methodology in several examples with European options involving up to 32 underlying assets.

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