A Multi-Level Monte-Carlo with FEM for XVA in European Options

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Abstract Counterparty credit risk has been recently incorporated in the pricing of financial derivatives by adding different adjustments, the set of which is referred as XVA. In the case of European options to consider stochastic default intensities, instead of constant ones, a three factor model arises. In this work, we have combined a numerical method for solving PDEs with Monte Carlo based techniques, to solve a new hybrid model for XVA pricing. In this way, instead of solving a three dimensional PDEs problem we solve a one dimensional PDE, with two stochastic coefficients coming from the stochastic intensities. More specifically, we propose the use of a Multi-Level Monte Carlo method.

1 Introduction

After Credit Crisis in 2008, unexpected defaults of big companies increased the relevance of counterparty risk in industry and academia. In derivative contracts, counterparty risk refers to the possibility that a counterparty defaults while owing money associated to the contract or while the mark-to-market value of the derivative is positive for the other part of the contract. Many papers and books developed techniques for the valuation of derivatives including counterparty risk by means of valuation adjustments, the set of all of them being referred as total valued adjustment and denoted by XVA. Some particular adjustments included in XVA are:

- CVA: the cost of hedging counterparty credit risk;
- DVA: the adjustment to a derivative price due to the institution's own default risk;

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- FVA: the correction made to the derivative price to account for a funding cost/benefit related to counterparty risk;
- KVA: the cost of holding regulatory capital associated to counterparty risk.

In order to compute the derivative value including the XVA or the price of the XVA, three main approaches are considered in the literature: partial differential equations (PDEs), backward stochastic differential equations (BSDEs) and formulations in terms of expectations. In the PDEs based approach, the spatial dimension of the time dependent PDE is equal to the number of underlying stochastic factors. In many settings, like pricing basket options or interest rate derivatives depending on a large number of forward or swap rates (LIBOR models), the required number of stochastic factors to develop a realistic pricing implies a PDE with high dimension, thus leading to the so called *curse of dimensionality* when numerical methods are addressed. In the present work, in order to overcome the curse of dimensionality, we aim to exploit the combination of Monte Carlo methods and the numerical solution of PDEs with one spatial dimension following the ideas in [5].

2 Modelling with Constant Intensities

Following [3], we consider a derivative contract between two defaultable parties, the hedger (H) and the investor (I). In order to obtain the value of the derivative including counterparty risk, the authors consider a portfolio with four traded assets:

- P^R : default risk-free, zero-coupon bond, with yield r ;
- P^H: default risky, zero-recovery, zero-coupon bond of party H, with yield r^H ;
- P^I : default risky, zero-recovery, zero-coupon bond of party I, with yield r^I ;
- S : underlying asset with no default risk.

Different linear and nonlinear PDEs formulations of the pricing problem can be obtained. The type of PDE depends on the choice of the so called mark to market (MtM) close outs, which is the value of the derivative in case of default. More precisely, let $\hat{V}_t$ be the value of the derivative with counterparty risk (hereafter referred as *risky derivative*), and let V_t be the value of the derivative without counterparty risk (risk free derivative). Possible choices to model the MtM value can be $M_t = \hat{V}_t$ (i.e., equal to the risky derivative value) or $M_t = V_t$ (i.e., equal to the risk free derivative value). In any case, the value of the risky derivative in case of default is:

- $\hat{V}_t = M^+(t,S) + R^I M^-(t,S)$, if the investor I defaults first;
- $\hat{V}_t = M^-(t,S) + R^H M^+(t,S)$, if the hedger H defaults first,

$R^I \in [0,1]$ and $R^H \in [0,1]$ being the recovery rates of parties I and H, respectively.

By using dynamic hedging methodology and different versions of Ito lemma, according to the choice of M, two different PDEs arise.

– Non Linear PDE when $M_t = \hat{V}_t$:

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}\hat{V} - r\hat{V} = (1 - R^H)\lambda^H (\hat{V})^- + (1 - R^I)\lambda^I (\hat{V})^+ + s^F (\hat{V})^+, \\ \hat{V}(T, S) = H(S). \end{cases} \tag{1}$$

– Linear PDE when $M_t = V_t$:

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}\hat{V} - (r + \lambda^H + \lambda^I)\hat{V} = -(R^H\lambda^H + \lambda^I)V^- - (R^I\lambda^I + \lambda^H)V^+ + s^F (V)^+, \\ \hat{V}(T, S) = H(S), \end{cases} \tag{2}$$

where we use the differential operator $\mathcal{A} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + r_R S \frac{\partial}{\partial S}$ and the notation x^+ and x^- for the positive and negative parts of x. Moreover, r_R is the rate paid in a repurchase agreement, s^F is the funding cost, r^F is the hedger funding rate, λ^H and λ^I denote the constant intensities of default of hedger and investor, respectively. The function H is the pay-off of the derivative in terms of the underlying asset price S.

If U denotes the XVA, then $\hat{V} = V + U$, where V is the price of the risk-free derivative. For a European vanilla option, the Black-Scholes formula provides the value of V. From (1) and (2), we obtain the corresponding PDEs for the XVA price.

– If $M_t = \hat{V}_t$ then U satisfies the nonlinear PDE problem:

$$\begin{cases} \partial_t U + \mathcal{A}U - rU = (1 - R^H)\lambda^H (V + U)^- + (1 - R^I)\lambda^I (V + U)^+ + s^F (V + U)^+, \\ U(T, S) = 0; \end{cases} \tag{3}$$

– If $M_t = V_t$ then U satisfies the linear PDE problem:

$$\begin{cases} \partial_t U + \mathcal{A}U - (r + \lambda^H + \lambda^I)U = (1 - R^H)\lambda^H (V)^- + (1 - R^I)\lambda^I (V)^+ + s^F (V)^+, \\ U(T, S) = 0, \end{cases} \tag{4}$$

Note that in the case of constant intensities of default, there is only one stochastic factor S_t and the spatial dimension of the governing PDE is equal to one. PDEs problems (3) and (4) for constant intensities have been numerically solved in [1], where the method of characteristics (Semi-Lagrangian method) for time discretization is combined with a finite element method (FEM) for the spatial discretization. Additionally, a fixed point iteration is applied to solve the nonlinear PDE.

3 Hybrid Models for Stochastic Intensities

The main objective of the present work is the extension of the previous setting with constant intensities of default to the case with stochastic intensities of default. For this purpose, we pose a hybrid model with three stochastic factors, which is governed by a PDE with two coefficients that are stochastic factors. This approach avoids the alternative consideration of a PDE with three spatial variables, the numerical solution of which is more computationally demanding. Thus, we pose the following linear PDE with one spatial dimension and the two stochastic coefficients:

$$\begin{cases} \partial_t U + \mathcal{A}U - (r + \lambda_t^H + \lambda_t^I)U = (1 - R^H)\lambda_t^H (V)^- + (1 - R^I)\lambda_t^I (V)^+ + s^F (V)^+, \\ U(T, S) = 0, \end{cases}$$

where the stochastic default intensities satisfy the following SDEs:

$$d\lambda_t^I = -\frac{k^I}{1 - R^I}\lambda_t^I dt + \frac{\sigma^I}{1 - R^I} dW_t^I, \quad d\lambda_t^H = -\frac{k^H}{1 - R^H}\lambda_t^H dt + \frac{\sigma^H}{1 - R^H} dW_t^H,$$

with σ^I and σ^H being the volatilities of the intensities of default while k^I and k^H are drift parameters. W_t^I and W_t^H are Brownian motions.

4 Numerical Methods for the Hybrid Model

A first possible naive approach to solve the hybrid model consists in using a crude Monte Carlo (MC) to simulate the paths of the stochastic intensities at the discrete times of the time discretization mesh used for the PDE numerical solution. This method can be sketched as follows:

- Simulate N paths of λ^H and λ^I (i.e., $\lambda^{H,i}$ and $\lambda^{I,i}$, $i = 1, \ldots, N$.)
- Solve numerically the (linear or nonlinear) PDE for each path to obtain $\tilde{U}_i$.
- Compute, as solution of the model, the expectation by using Monte Carlo with:

$$\mathbb{E}[\tilde{U}] = \frac{1}{N}\sum_{i=1}^{N} \tilde{U}_i$$

In the present work, we also aim to speed up the Monte Carlo convergence and reduce the variance by using the Multi Level Monte Carlo (MLMC) method presented in [4]. The main ideas of MLMC can be summarized as follows:
If we want to compute the expected value of a process $Q = F(S_t)$, where the process satisfies $dS_t = a(S_t, t)dt + b(S_t, t)dW_t$ and $t \in [0, T]$, we can write:

$$\mathbb{E}[\hat{Q}_L] = \mathbb{E}[\hat{Q}_0] + \sum_{l=1}^{L} \mathbb{E}[\hat{Q}_l - \hat{Q}_{l-1}],$$

where $L > 0$ is a positive integer and $\hat{Q}_l$ is an approximation of Q, estimated on the discretisation of the time interval with the time step $h_l = \frac{T}{M^l}$, M being a positive integer. Let Y_l denote an approximation of $\mathbb{E}[\hat{Q}_l]$, then:

$$Y_L = Y_0 + \sum_{l=1}^{L} Y_l - Y_{l-1}.$$

Therefore, each Y_l is computed with the MC method, using N_l simulations.

5 Numerical Results

We consider an example with a European put option and we compare the case of constant intensities (1-factor model) with a couple of cases with stochastic intensities (2-factor and 3-factor models). We use a linear PDE model, which is numerically solved with the method developed in [1] using a uniform spatial mesh with 1000 nodes and a time step depending on the level in the MLMC method. As MLMC parameters we consider $L = 4$ and $M = 4$, with $N_l = 500$ simulations per level.

First, assuming that $\lambda^I = 0$ we compare the 1-factor and 2-factor models corresponding to the cases λ^H constant and λ^H stochastic, respectively. The values of the parameters are $\sigma = 0.3$, $r = 0.04$, $\lambda_0^H = 0.04$ (constant case and initial intensity in stochastic case), $R^H = 0.4$, $R^I = 0.3$, $k^H \in \{0.1, 0.3, 0.5, 0.7\}$, $\sigma^H = 0.2$, the strike $K = 2$ and the maturity $T = 0.5$. The PDE variables are $t \in [0, T]$ and $S \in [0, 3]$. Next, we compare the 1-factor and 3-factor models, where we additionally consider a stochastic λ^I, with parameters $\lambda_0^I = 0.04$, $k^I \in \{0.1, 0.3, 0.5, 0.7\}$ and $\sigma^I = 0.2$.

In Figure 1 we show the XVA prices of 1-factor versus 2-factor (left) models and of 1-factor versus 3-factor models (right), illustrating that differences increase for small values of the underlying asset and for larger values of drift coefficients in the stochastic equations governing intensities of default. The XVA is negative as it represents the decrease in the risk free put value due to the probability of default. For small values of the asset, the put option is in the money and one counterparty will be interested in exercising so he/she will be (more) exposed to the other counterparty default. As the exposure has a more negative impact on the put option value for smaller asset values, the XVA becomes more negative and more sensitive to the variation of the drifts of the stochastic intensities of default.

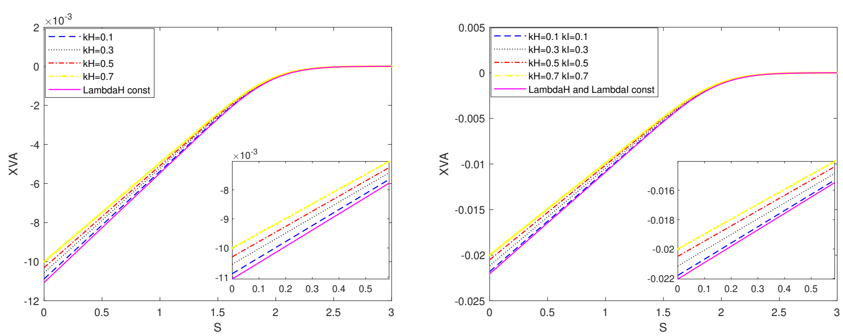

Fig. 1: Comparison 1-factor versus 2-factor models (left) and 1-factor versus 3-factor models (right), for different drifts in stochastic intensities.

Finally, by using as reference solution the one obtained with MLMC with parameters $L = 5$, $N_l = 2000$, we compare the crude MC and the MLMC for the 3-factors hybrid model. Figure 2 shows the maximum error with respect to time step (left) and computational times (right), clearly illustrating the advantages of MLMC.

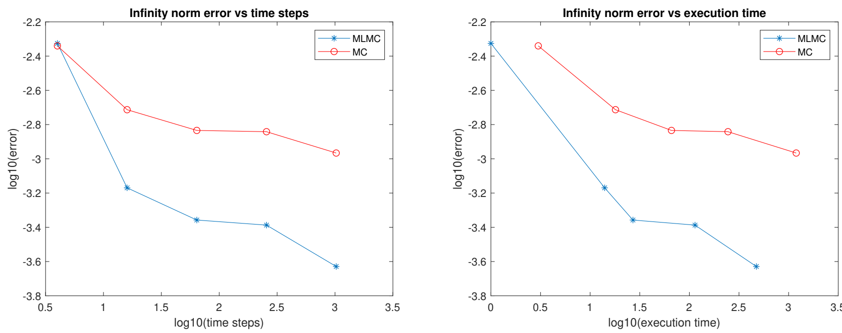

Fig. 2: Comparison of errors in crude Monte Carlo (MC) and Multi Level Monte Carlo (MLMC).

6 Conclusions

A hybrid model has been proposed for the case of stochastic intensities of default involving three factors in the evaluation of XVA. The hybrid approach allows to consider PDEs with one spatial dimension and two stochastic coefficients, thus avoiding the solution for PDEs with three spatial dimensions. Multi Level Monte Carlo speeds up the convergences with respect to the use of a crude Monte Carlo numerical methodology. Numerical results illustrate the effect of considering more realistic stochastic intensities of default with respect to constant ones. More details, specially about the numerical examples and their discussion, will appear in [2].

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