Industrial problem 2: SABR model and Hagan's formula

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UNIVERSIDADE DA CORUÑA

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Hagan's formula *H*

$$\text{SABR model parameters} \xrightarrow{H} \text{implied volatilities}$$

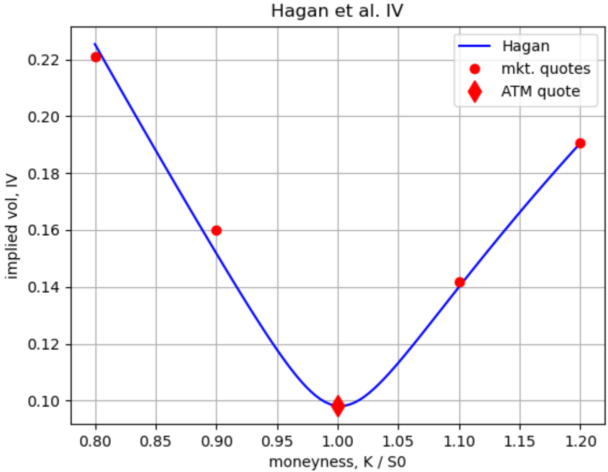

Figure: Calibration of the model to market prices with the formula

Butterfly arbitrage is present in Hagan's formula

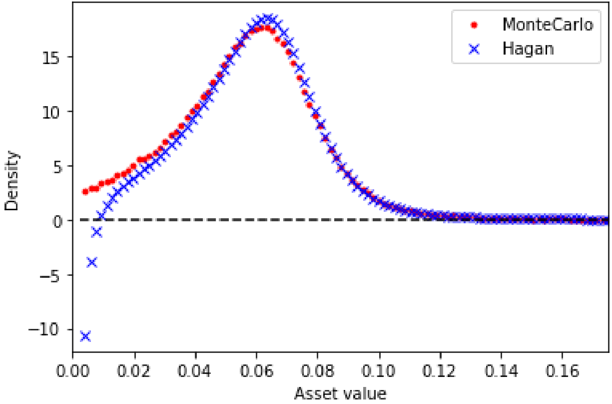

Figure: Comparison between the density from the SABR implied volatility formula and the MC density. Negative values imply butterfly arbitrage.

Hybrid ML–H extension

1. Neural network learns error of Hagan's formula by comparison to MC.
2. In addition: learns error of the Greek $\partial C/\partial K$ implied by Hagan's formula and implied by MC simulation. (Exact! Automatic differentiation!).
3. More Greeks to be added, $\partial^2 C/\partial K^2$ implicitly solves the butterfly arbitrage problem.

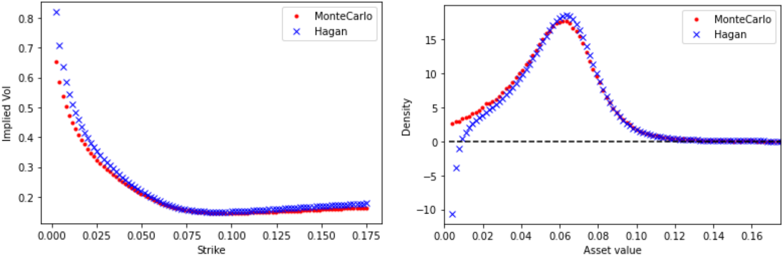

Figure: The NN learns to fix both errors: Difference between IV's (left) and densities (right)

SABR-CIR extension

$$\begin{aligned} dS_t &= v_t S_t^\beta dW_t^1, & S_0 &= S \\ dv_t &= \gamma v_t dW_t^2, & v_0 &= \alpha \\ dW_t^1 dW_t^2 &= \rho dt \end{aligned}$$

The SABR model has limitations, e.g. $\mathbb{E}[v_t] \equiv \alpha$, so no volatility term structure can be fit.

Proposed extension: SABR-CIR model

$$\begin{aligned} dS_t &= u_t S_t^\beta dW_t^1, & S_0 &= S \\ du_t &= \kappa(\bar{u} - u_t)dt + \sigma\sqrt{u_t}dB_t, & u_0 &= u0 \\ dW_t^1 dB_t &= \rho dt \end{aligned}$$

SABR-CIR calibration with Hagan's formula

1. Let $\mathscr{S}$ be the SABR parameters and let $\mathscr{C}$ be the SABR-CIR parameters.
2. We relate these parameters by matching of integrated process moments

$$\mathbb{E}[\int_0^T v_t dt] \stackrel{!}{=} \mathbb{E}[\int_0^T u_t dt] \tag{1}$$

$$\mathbb{E}[(\int_0^T v_t dt)^2] \stackrel{!}{=} \mathbb{E}[(\int_0^T u_t dt)^2] \tag{2}$$

3. We obtain synthetic SABR parameters $\mathscr{S} = f(\mathscr{C})$.
4. Evaluation of Hagen's formula let's us calibrate the model semi-analytically[1]:
   Match $H\big(f(\mathscr{C})\big)$ to market

[1] the functional relationship f depends on a fast numerical optimization

Application of hybrid ML–H extension

Hagan's formula evaluated at the synthetic parameters $f(\mathscr{C})$ produces a larger error across the entire domain. This acts as a control variate to which we apply the previous ML model.

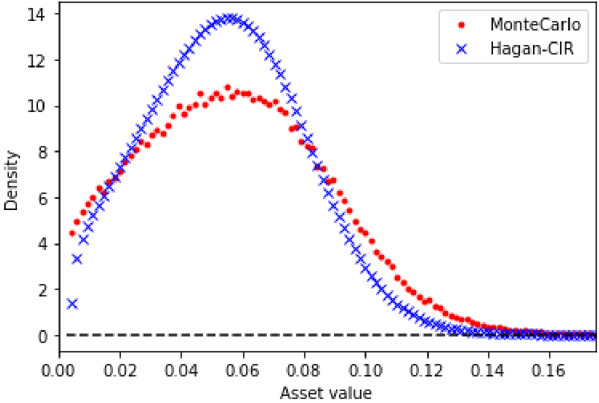

Figure: Comparison between the density from the SABR-CIR implied volatility formula and the actual density