Modelling jump clustering from a Pólya process with limited memory

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Motivation

- Cluster of defaults on a given portfolio
- Jumps of an asset or index due to the successive arrival of news.
- Claims due to accidents or catastrophes.

Default clustering

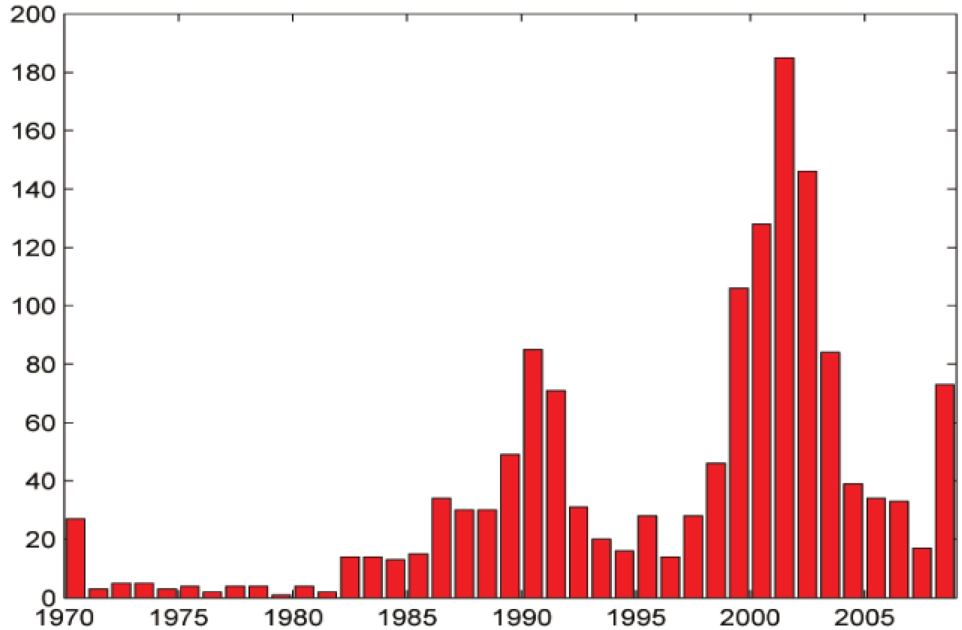
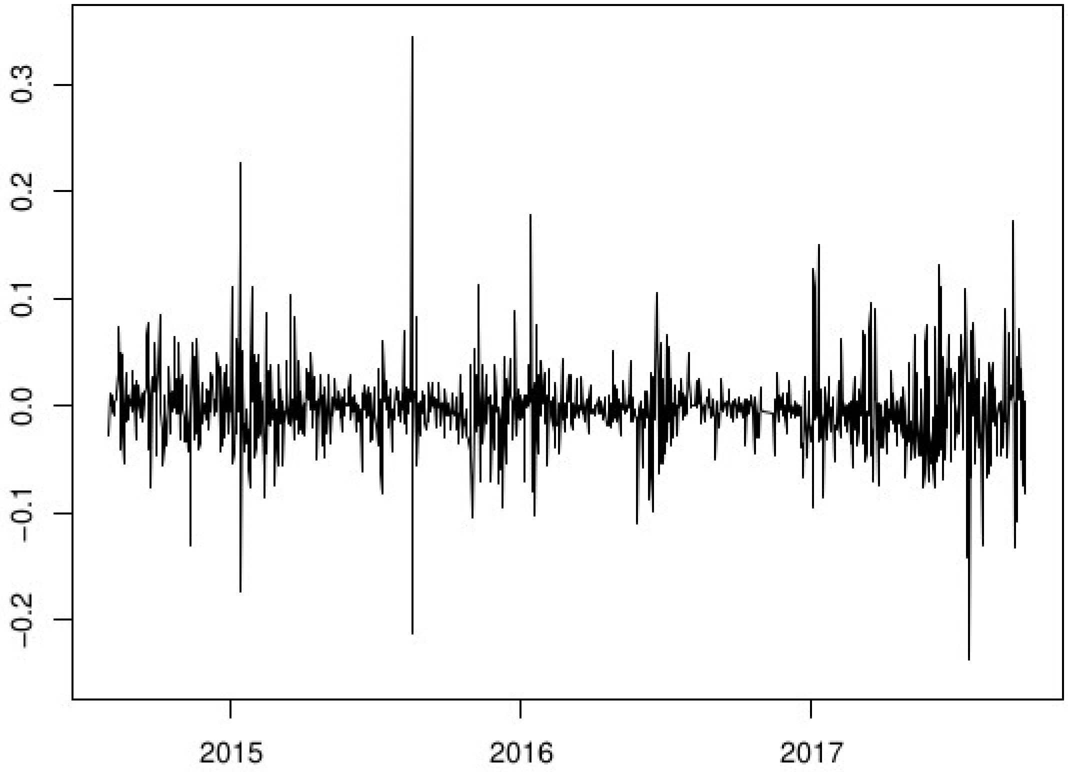


Figure 1. *Annual defaults of Moody's rated U.S. firms between January* 1970 *and November* 17, 2008. *Source: Moody's Default Risk Service. The peak in* 1970 *represents a cluster of* 24 *railway defaults triggered by the collapse of Penn Central Railway on June* 21, 1970. *The fallout of the* 1987 *crash is indicated by the peak in the early* 1990*s. The burst of the internet bubble caused many defaults during* 2001–2002. *From a trough in* 2007*, default rates increased significantly in* 2008.

Source: Errais et al. (2010). Affine Point Processes and Portfolio Credit Risk. SIAM J. Financial Math. Vol. 1, pp. 642-665.

Bitcoin returns



Source: Chen, C. Y., Härdle, W. K., Hou, A. J., & Wang, W. (2018). Pricing Cryptocurrency Options: The Case of CRIX and Bitcoin. SSRN Electronic Journal . doi:10.2139/ssrn.3159130

Arrival of claims

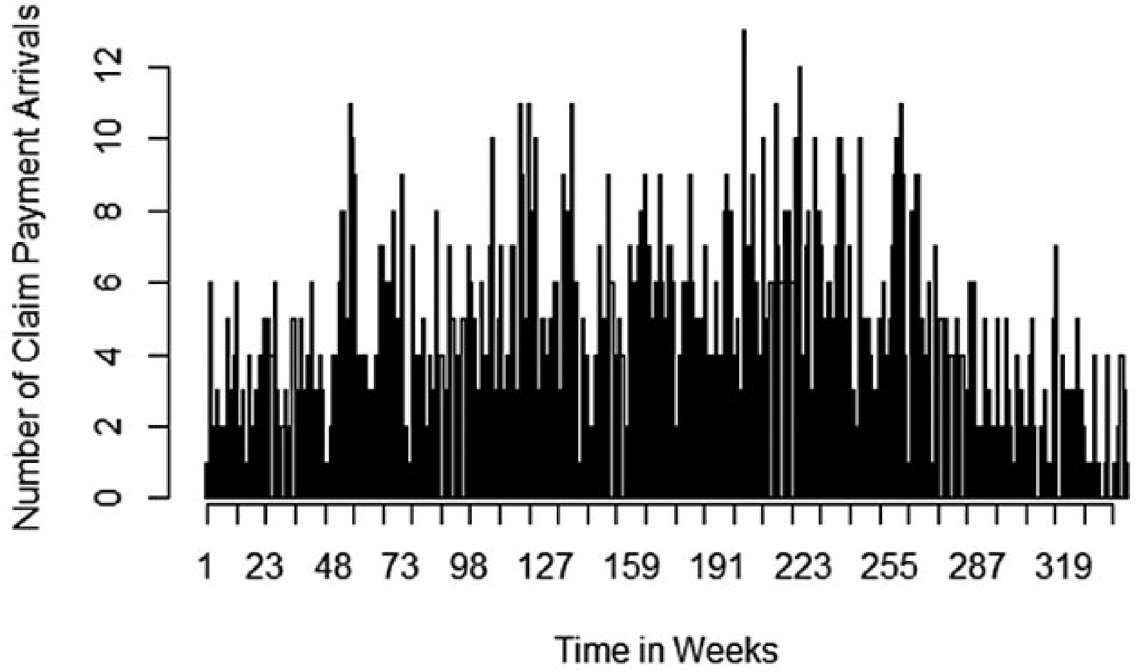


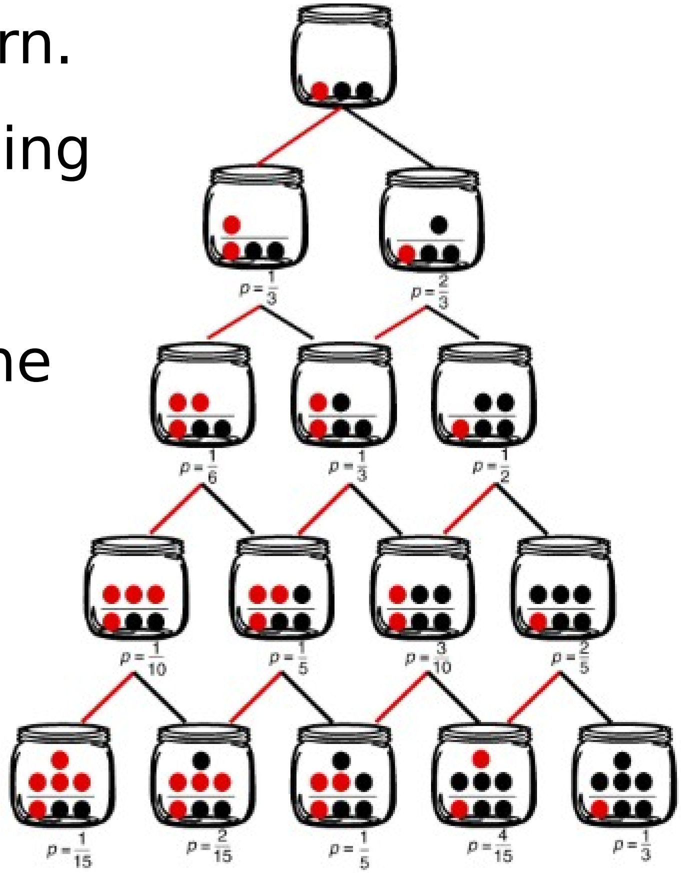
Fig. 2. The number of claim payments per week during the time period 01 Jan 2010 to 28 July 2016 gives further indication as to the presence of clustering.

Source: Swishchuk et al (2021). Hawkes processes in insurance: Risk model, application to empirical data and optimal investment. Insurance, Mathematics and Economics.

Urn model: the Pólya process

- At each time we select a ball from the urn.
- Each ball has the same probability of being chosen.
- Add an extra ball of the same color as the one selected.
- Represented by the schema:

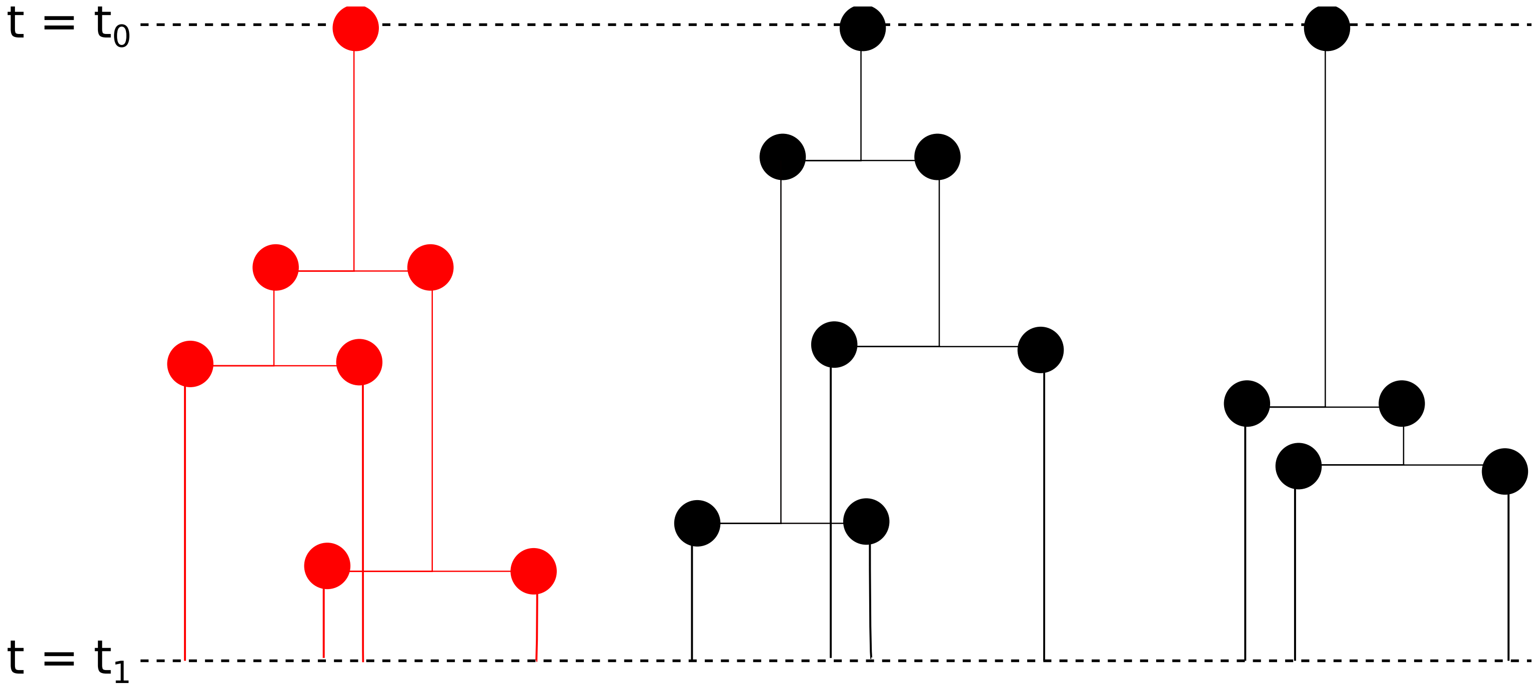
$$A = \begin{pmatrix} {\color{red}r} & 0 \\ {\color{red}0} & r \end{pmatrix}$$



Poissonization of urns

- The previous urn model only works in discrete times.

- To make it time-continuous we attach an exponential clock to every ball.



SDE for Poissonized Pólya

- Being R_t the number of red balls at time *t*,

$$dR_t = r \cdot X_t, \quad X_t \sim Poi(R_{t\_}\, dt)$$

- The intensity is proportional to the process itself, making it *self-exciting.*

Analytical properties

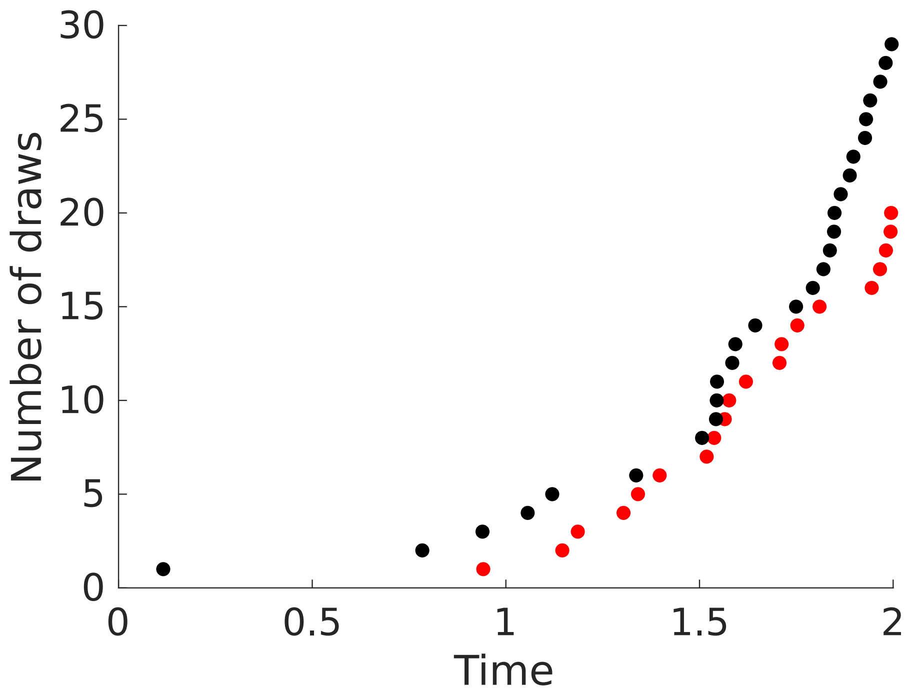
- Characteristic function:

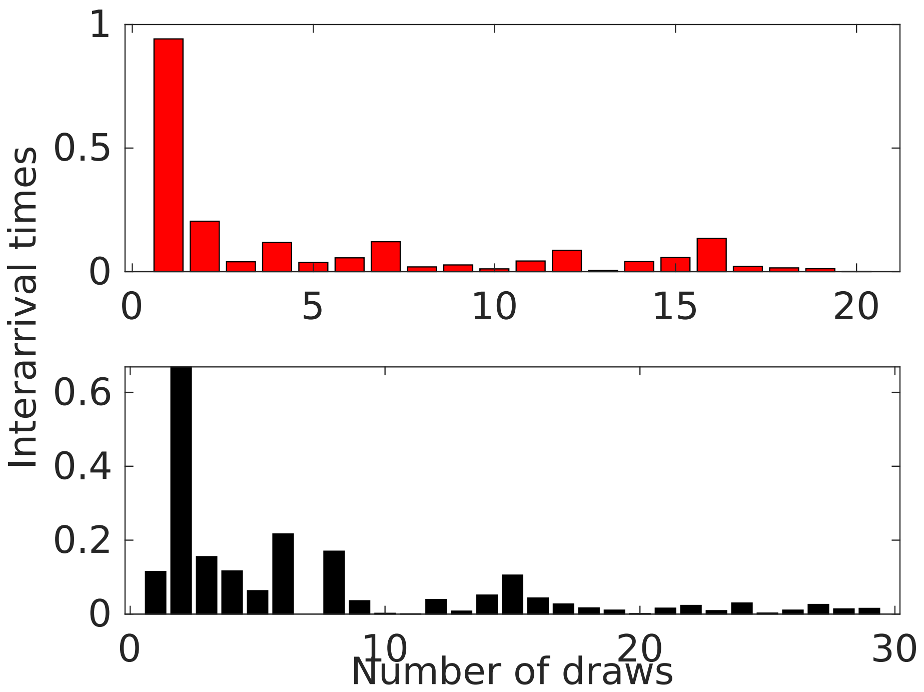
$$\mathbb{E}[e^{iuR_t}] = \left(\frac{e^{-rt} e^{iur}}{1 - (1 - e^{-rt}) e^{iur}} \right)^{\frac{R_0}{r}} .$$

- Density function:

$$\mathbb{P}[R_t = R_0 + r \cdot k] = \binom{k + \frac{R_0}{r} - 1}{k} p^{R_0} (1 - p^r)^k \quad k = 0, 1, ...,$$

The exploding Pólya





Urns with memory kernel

We attach a memory kernel to each **new** ball, reducing its importance over time. Some properties of this kernel are:

- $\phi(t = 0) = 1$
- $\phi(t \to \infty) = 0$
- Non-increasing function of time.
- Examples:

$$\phi_1 = e^{-\lambda t}, \qquad \phi_2 = \mathbb{1}_{\{t \leq \Delta\}},$$

Adapting the Pólya SDE

Given that

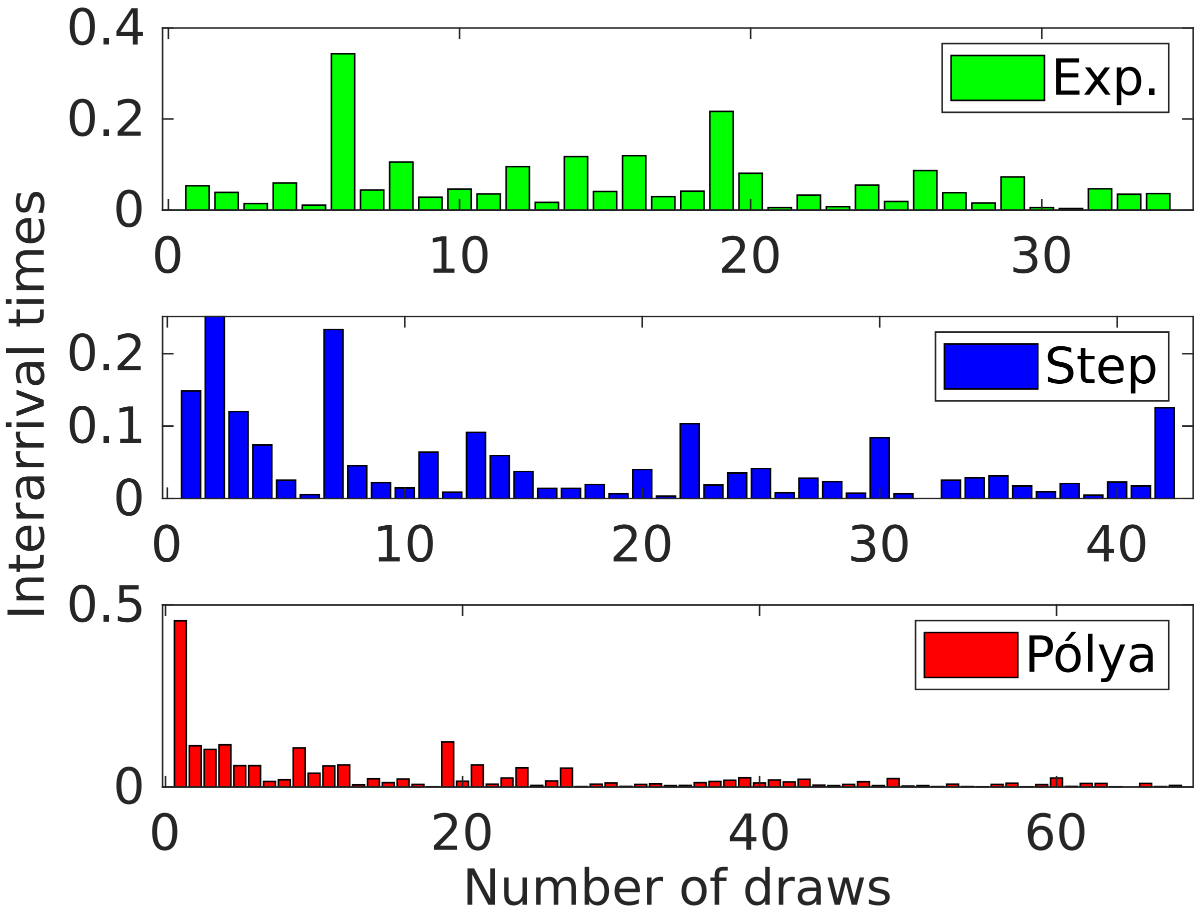
$$R_{t_-} = R_0 + \int_0^{t_-} dR_s,$$

we get the SDE for the Pólya process with limited memory:

$$dR_t = r \cdot X_t, \quad X_t \sim Poi(\lambda_t dt),$$

$$\lambda_t = R_0 + \int_0^{t_-} \phi(t-s) dR_s.$$

New interarrival times



The Hawkes process

- It turns out the Pólya process is a particular case of the Hawkes process

$$\lambda_t = \mu(t) + \int_0^{t_-} \phi(t-s)\, dN_s.$$

- Hopefully we can use some of its properties to efficiently simulate the Pólya process.

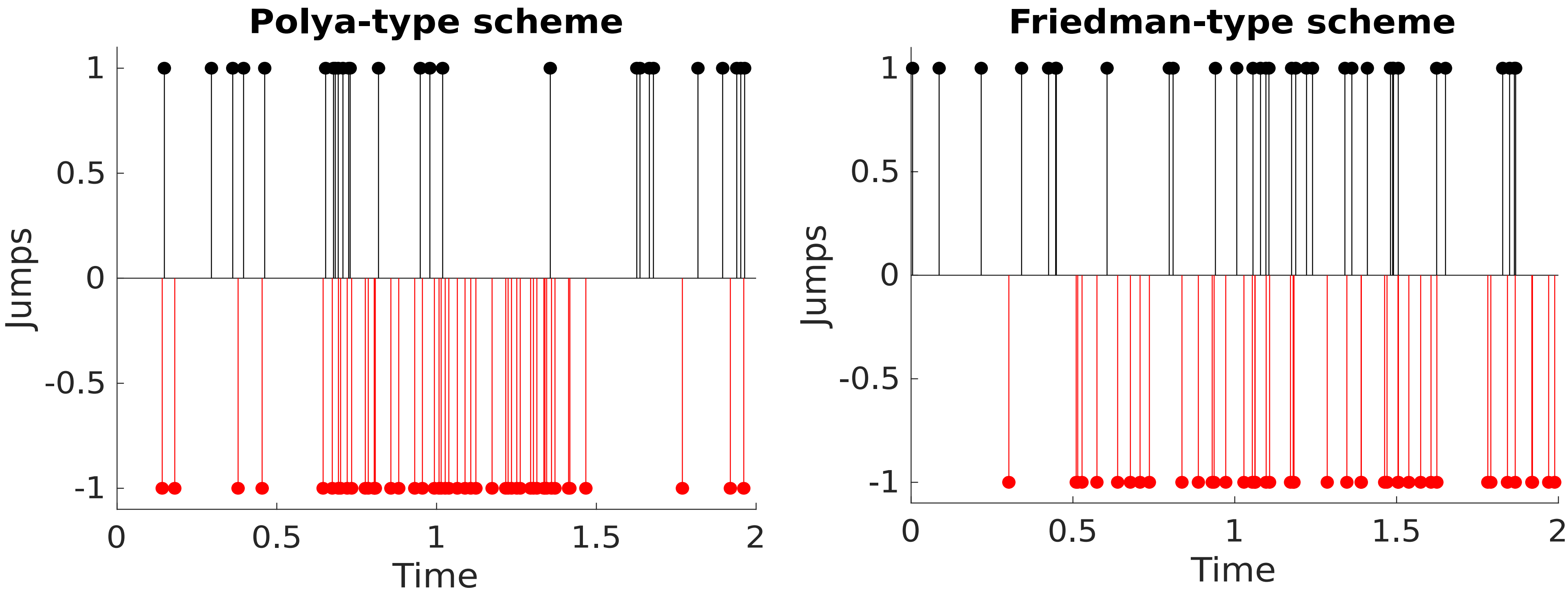
- For example, the characteristic function can be computed via a delayed differential equation.

Where does the jump go?

- Also important to analyze the direction of the next jump. Is it **downward** or **upward**?
- We can do this associating a color to each direction.
- The Pólya model favours jumps in the **same** direction.
- We can favour jumps in the **opposite** direction using the Friedman urn model:

$$A = \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix}$$

Pólya vs. Friedman



Future work

- Study the connection between Poissonized urn models and the Hawkes process.

- Use of the combinatorial properties of urns to improve numerical algorithms for computation of moments, characteristic function, etc.

- Analyze the performance of the generalized Pólya-Friedman model in actual applications.

Questions?

Thanks for your attention!