Deep learning of implied volatility surfaces in time

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Implied volatility surfaces

- Traders use implied volatilities in option pricing to determine if they believe a financial product is under- or overpriced.
- Inversion of the Black-Scholes model will give the volatility, if we know the option value.
- An implied volatility surface defines a 3-dimensional plot.
- The question arises how such a volatility surface moves over time and when a recalibration of the model will be required.
- Demand for accurate and robust techniques, as implied volatility shapes differ much over time and among different companies.

The focus of this project

- How to implement neural networks for this task, understand them, and find good metrics to compare the output of these models.
- Option data from the S&P 500 index of October 2020/March 2022 available to construct volatility smiles and volatility surfaces.
- Initially, apply neural networks on one specific trading day to understand the models before expanding to a multiple days data set.
- We don't have many strikes and maturities available, and they are unevenly distributed among the strike and time axis with more quotes for strikes at the money and short time to maturity options.

Before using the data, some operations may be applied

- Some inaccurate quotes may be filtered out, as they may relate to noisy data. A number of features, as the log-moneyness and mid-price could be calculated and added to the data set.
- We lack two main input values to use the Black-Scholes formula: interest rate and the index's price. The put-call parity may help.

  Reference: Yu Zheng, Yongxin Yang and Bowei Chen, *Incorporating Prior Financial Domain Knowledge into Neural Networks for Implied Volatility Surface Prediction.* https://arxiv.org/pdf/1904.12834.pdf

Traders have opinion of how a fit should look to be useful

- Log-moneyness values are around 0, should be close to perfect.
- It is important to bring financial conditions, that have to be met in practice, into a neural network, while, at the same time, we require a flexible technique to represent many different implied volatility shapes.
- Let

$$d_{\pm}(m, T) = -\frac{m}{\sqrt{T}\sigma_{BS}} \pm \frac{1}{2}\sqrt{T}\sigma_{BS}(m, T),$$

  $n(\cdot)$ is standard normal density, $N(\cdot)$ the corresponding distribution, T time to expiration, m log-moneyness and σ_{BS} is implied volatility.

Conditions for the implied volatility surface

- Positivity: For $(m, T) \in \mathbb{R} \times \mathbb{R}^+, \ \sigma_{BS}(m, T) > 0$;
- For $T > 0$, function $m \to \sigma_{BS}(m, T)$ is twice differentiable on $\mathbb{R}$.
- Monotonicity: For $m \in \mathbb{R}, \ T \to \sqrt{T}\sigma_{BS}(m, T)$ is increasing on $\mathbb{R}^+$, and

$$\sigma_{BS}(m, T) + 2T\frac{\partial \sigma_{BS}(m, T)}{\partial T} \geq 0.$$

- Limit condition: If $T > 0$, then $\lim_{k\to\infty} d_+(m, T) = -\infty$
- Asymptotic slope: For $T > 0$, then $2|m| - \sigma^2_{BS}(m, T)T > 0$

Research past and present

- Butterfly arbitrage-free: For $(m, T) \in \mathbb{R} \times \mathbb{R}^+$,

$$\left[1 - \frac{m\frac{\partial \sigma_{BS}(m,T)}{\partial m}}{\sigma_{BS}(m,T)}\right]^2 - \frac{1}{4}\left[\sigma_{BS}(m,T)T\frac{\partial \sigma_{BS}(m,T)}{\partial m}\right]^2 + T\sigma_{BS}(m,T)\frac{\partial^2 \sigma_{BS}(m,T)}{\partial m^2} \geq 0$$

- Right boundary: If $m \geq 0$, then

$$N(d_-(m,T)) - \sqrt{T}\frac{\partial \sigma_{BS}(m,T)}{\partial m}n(d_-(m,T)) \geq 0$$

- Left boundary: If $m < 0$, then

$$N(-d_-(m,T)) + \sqrt{T}\frac{\partial \sigma_{BS}(m,T)}{\partial m}n(d_-(m,T)) \geq 0$$