On Fixing and Calibrating of the Volatility Formulas

Problem Specification

1 Introduction

When handling many market volatility quotes, it is natural to express them in terms of some parametric form so that only a few parameters can explain a whole range of strikes. Moreover, once the parametric equation is given, one can instantly obtain volatilities by evaluating the parametric function.

A market standard for volatility parameterization for several years, the well-known SABR model-based formula [4] originates from a short-maturity heat kernel expansion. Although very easy to implement, the density implied by the approximation is not always free of arbitrage, especially not for very low strikes (it becomes negative or the density does not integrate to one).

Over the last decade, several model improvements have been introduced, like in [1] whereby a one time-step finite difference approximation, the SABR model was solved or in [5] where the density was arbitrage-free but the method required the numerical solution of a probability density function. Other improvements on the density were introduced in [3] and [2]. Unfortunately, until this point, all the methods proposed to suffer from either expensive numerical methods involved or do not satisfy the arbitrage-free conditions. On the other hand, the calibration of the parametric formula to the market quotes is still a challenging task.

The objectives of this project are as follows:

1. Investigate "fixes" for the formula to mitigate the arbitrage opportunities.
2. When expressing the implied volatilities in terms of parametrized form (by using either the SVI or SABR-based formula), it is crucial to calibrate the parametric form to given market implied volatilities. In essence we need to determine the model parameters for which the distance of the model vs. market implied volatilities is the smallest, i.e., $||\sigma(T,\cdot) - \sigma_{market}(T,\cdot)|| \to 0$. The calibration procedure typically requires many iterations over many possible parameter configurations. This is considered to be an expensive task. The objective is to, utilizing ANN, learn the relation between market quotes and model parameters.

References

[1] J. Andreasen and B. Huge. Expanded forward volatility. *Risk*, pages 101–107, 2013.

[2] P. Balland and Q. Tran. SABR goes normal. *Risk*, pages 76–81, 2013.

[3] P. Doust. No-arbitrage SABR. *The Journal of Computational Finance*, 15(3):3–31, 2012.

[4] P.S. Hagan, D. Kumar, A.S. Leśniewski, and D.E Woodward. Managing smile risk. *Wilmott Magazine*, pages 84–108, 2002.

[5] P.S. Hagan, D. Kumar, A.S. Leśniewski, and D.E. Woodward. Arbitrage-free SABR. *Wilmott Magazine*, pages 60–75, 2014.

2 Implied volatility for the SABR model

The approximating implied volatility derived in [4] reads:

$$\sigma(T,K) = A(K)\frac{z(K)}{\chi(z(K))} + B(T,K), \quad \text{and} \quad \hat{\sigma}^H(T,S_0) = \frac{\alpha}{S_0^{1-\beta}}B(T,S_0),$$

where

$$\begin{aligned}
z(K) &= \frac{\gamma}{\alpha}(S_0K)^{(1-\beta)/2}\log(S_0/K), \quad \chi(z(K)) = \log\left(\frac{\sqrt{1-2\rho z(K)+z^2(K)}+z(K)-\rho}{1-\rho}\right),\\
A(K) &= \alpha\left(S_0K^{(1-\beta)/2}\left(1+\frac{(1-\beta)^2}{24}\log^2(S_0/K)+\frac{(1-\beta)^4}{1920}\log^4(S_0/K)\right)\right)^{-1},\\
B(T,K) &= \left\{1+\left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(S_0K)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\gamma\alpha}{(S_0K)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\gamma^2\right)\right\}T.
\end{aligned}$$