Industry Problem

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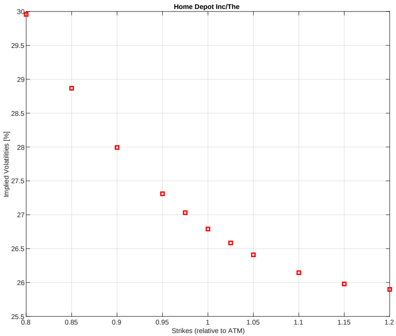
Overview

Implied Volatilities and Parametrizarions

- When handling many market volatility quotes, it is natural to express them in terms of some parametric form so that only a few parameters can explain a whole range of strikes. Moreover, once the parametric equation is given, one can instantly obtain volatilities by evaluating the parametric function.
- A market standard for volatility parameterization for several years, the well-known SABR model-based formula [4] originates from a short-maturity heat kernel expansion.

Implied Volatilities



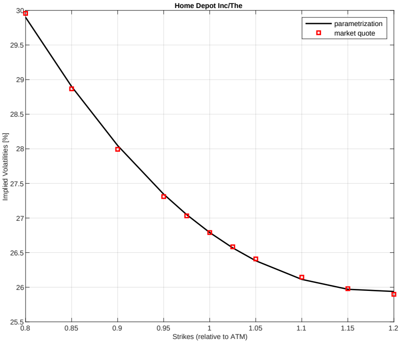


Figure: Left: Market Quotes; Right: Parameterization

The SABR model vs. the Heston model

- The Heston model:

$$\begin{aligned}\mathrm{d}S(t) &= \sqrt{v(t)}S(t)\mathrm{d}W_1(t),\\ \mathrm{d}v(t) &= \kappa\left(\bar{v}-v(t)\right)\mathrm{d}t+\gamma\sqrt{v(t)}\mathrm{d}W_2(t).\end{aligned}$$

- Model $\rightarrow$ FFT (COS method) $\rightarrow$ Option Price $\rightarrow$ Implied Volatilities
- The SABR model:

$$\begin{aligned}\mathrm{d}S(t) &= v(t)S^{\beta}(t)\mathrm{d}W_1(t),\\ \mathrm{d}v(t) &= \gamma v(t)\mathrm{d}W_2(t), \quad v(t_0)=\alpha.\end{aligned}$$

- Model $\rightarrow$ Implied Volatilities

Volatility Parametrizartions

The approximating implied volatility derived in [4] reads:

$$\sigma(T,K) = A(K)\frac{z(K)}{\chi(z(K))} + B(T,K),$$

where

$$\begin{aligned}
z(K) &= \frac{\gamma}{\alpha}(S_0K)^{(1-\beta)/2}\log(S_0/K),\\
\chi(z(K)) &= \log\left(\frac{\sqrt{1-2\rho z(K)+z^2(K)}+z(K)-\rho}{1-\rho}\right),\\
A(K) &= \alpha\left(S_0K^{(1-\beta)/2}\left(1+\frac{(1-\beta)^2}{24}\log^2(S_0/K)+\frac{(1-\beta)^4}{1920}\log^4(S_0/K)+\boxed{\epsilon}\right)\right)^{-1},\\
B(T,K) &= \left\{1+\left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(S_0K)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\gamma\alpha}{(S_0K)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\gamma^2\right)\right\}T,\\
\boxed{\epsilon} &= ???
\end{aligned}$$

Can $\boxed{\epsilon}$ be calibrated? By taking, e.g., $\epsilon = a_0 + a_1 S_0/K + a_2 S_0^2/K^2 + \ldots$?

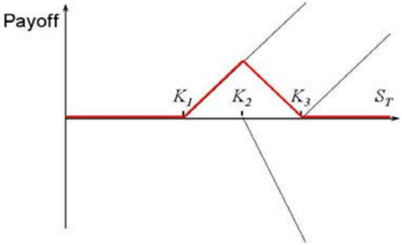
Calibration

- When expressing the implied volatilities in terms of parametrized form, it is crucial to calibrate the parametric form to given market implied volatilities.
- In essence we need to determine the model parameters for which the distance of the model vs. market implied volatilities is the smallest.
- The calibration procedure typically requires many iterations over many possible parameter configurations. This is considered to be an expensive task.
- The calibration of the SABR formula is always performed in two steps:
  - Model parameters are chosen, except for α.
  - α is chosen such that ATM volatilities are perfectly matched.

Types of arbitrage in the volatility objects

We distinguish two types of arbitrage in the volatility objects

- Calendar arbitrage: $C(T_1, K) > C(T_2, K)$, for $T_1 < T_2$ and where C is a call option and K is a strike.
- Butterfly arbitrage $C(T, K_1) - 2C(T, K_2) + C(T, K_3) < 0$ for $K_1 < K_2 < K_3$.



Butterfly Arbitrage

- Without loss of generality we can assume that $K_3 - K_2 = K_2 - K_1 =: \delta_K$ thus since $\delta_K > 0$ we have:

$$\frac{C(K+\delta_K) - 2C(K) + C(K-\delta_K)}{\delta_K^2} \approx \boxed{\frac{\partial^2 C(K)}{\partial K^2}} \quad \text{for} \quad \delta_K \to 0.$$

- A call price is given by:

$$C(K) = \int_{\mathbb{R}} \max(x-K,0) f_S(x) \mathrm{d}x,$$

so by differentiation we find the following relation:

$$\boxed{\frac{\partial^2 C(K)}{\partial K^2} = f_S(K)}.$$

- So the presence of the butterfly arbitrage is equivalent with assigning negative probabilities to stock's movements.
- The elimination of the butterfly arbitrage is equivalent with ensuring that probability density is nonnegative and it integrates to unit.

Arbitrage in the SABR's formula

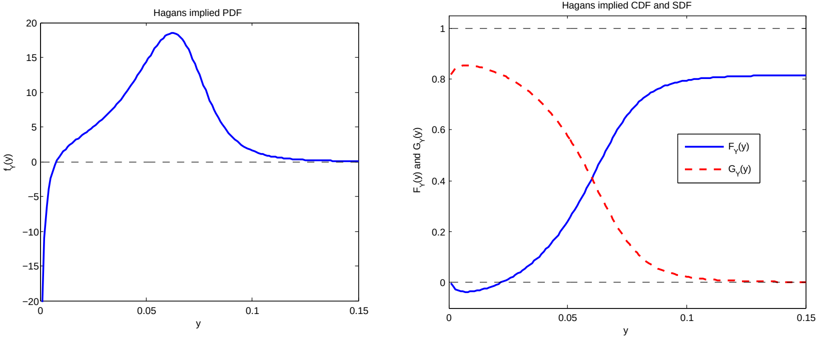


Figure: $\beta = 0.5$, $\alpha = 0.05$, $\rho = -0.7$, $\gamma = 0.4$, $F(t_0) = 0.05$ and $T = 7$. Left: probability density, with deterioration near zero; right: corresponding CDF and SDF (survival distribution function).

Research Questions

The objectives of this project are as follows:

1. Develop a two stage ANN calibration algorithm for the calibration of the SABR model formula.
2. Investigate "fixes", $\boxed{\epsilon}$, for the formula to mitigate the arbitrage opportunities [1, 5, 3, 2].

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