Stochastic Modelling of the Collateral Choice Option and its Practical Implications

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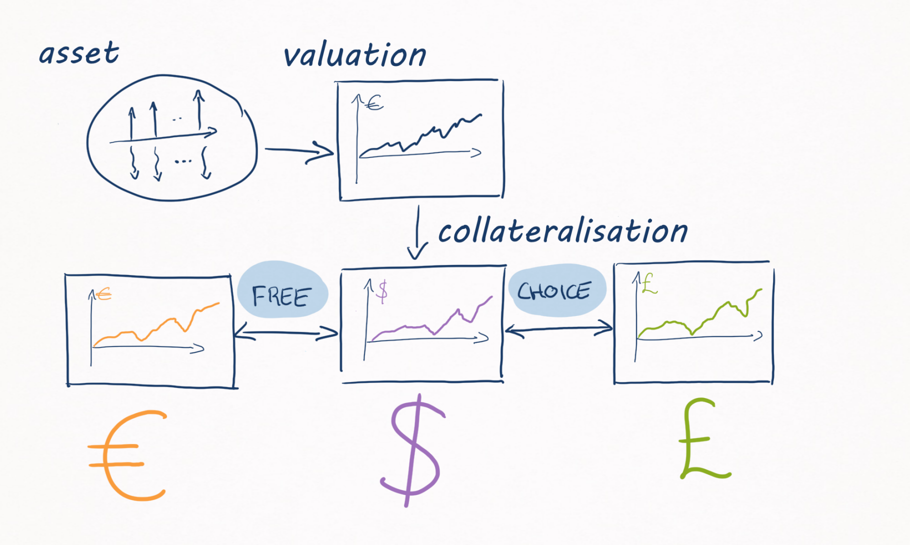
The collateral choice option



The collateral choice option

- Assets are perfectly collateralized:
  no frictions, entire asset is covered at all times.

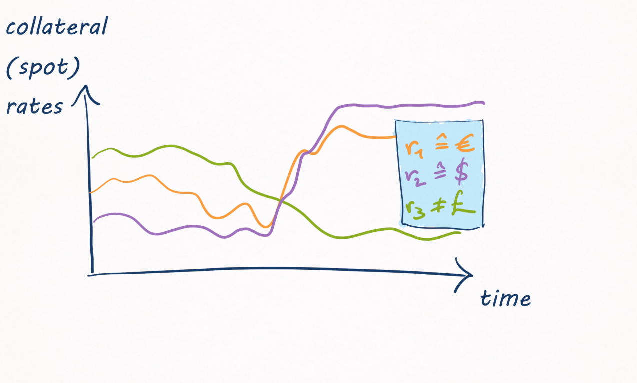
- No default risk, discounting with time value of money.

- Single curve framework based on *collateral rate r*.

- Asset valuation becomes

$$V_0 = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t} V_T\right].$$

The collateral choice option



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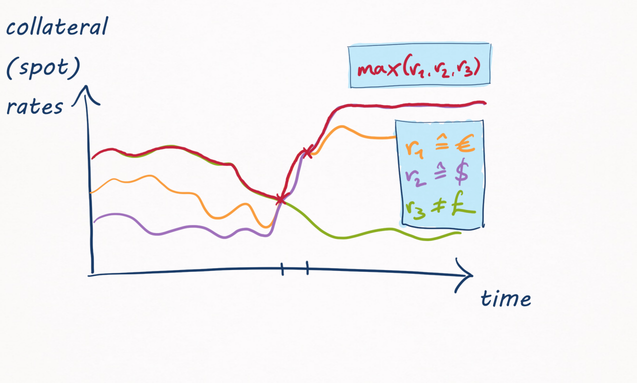
The collateral choice option



$$V_0 = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T \max(r_0(t),\ldots,r_N(t))\mathrm{d}t} V_T\right]$$

The CTD discount factor

Goal: Express the option without path and asset dependence.

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T \max(r_0(t),\dots,r_N(t))\mathrm{d}t\right)V_T\right]$$

$$=\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_0(t)+\max(0,(r_1-r_0)(t),\dots,(r_N-r_0)(t))\mathrm{d}t\right)V_T\right]$$

$$=\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T \max(0,q_1(t),\dots,q_N(t))\mathrm{d}t\right)\exp\left(-\int_0^T r_0(t)\mathrm{d}t\right)V_T\right]$$

The CTD discount factor

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Strong independence (all q_i from r_0 and V):

$$= \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T \max(0,q_1(t),\ldots,q_N(t))\mathrm{d}t}\right]\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T r_0(t)\mathrm{d}t} V_T\right].$$

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Weak independence (all q_i from V):

$$= \mathbb{E}^{\mathbb{Q}^T}\left[\mathrm{e}^{-\int_0^T \max(0,q_1(t),\dots,q_N(t))\mathrm{d}t}\right]\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T r_0(t)\mathrm{d}t} V_T\right].$$

The CTD discount factor

For either measure in $\{\mathbb{Q}, \mathbb{Q}^T\}$:

$$\mathrm{CTD}(0, T) = \mathbb{E}\left[\mathrm{e}^{-\int_0^T \max(0, q_1(t), \ldots, q_N(t))\mathrm{d}t}\right].$$

Hard to evaluate (in general)!

The CTD discount factor

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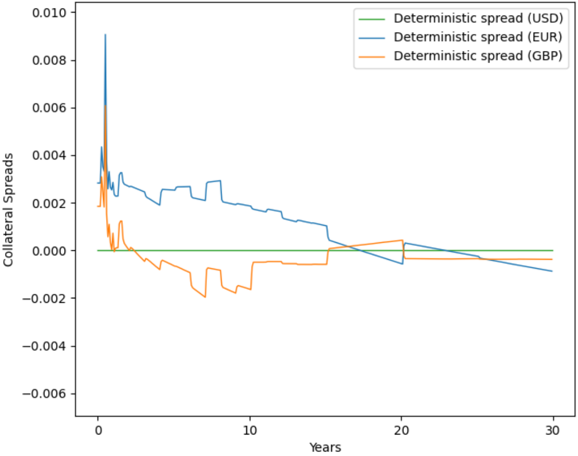
Hard to evaluate (in general)!

Pragmatic solution: deterministic model

$$\mathrm{CTD}_{\mathrm{det}}(0, T) = \mathrm{e}^{-\int_0^T \max(0, \mathbb{E}[q_1(t)], \dots, \mathbb{E}[q_N(t)])\mathrm{d}t}.$$

The model choice matters

The deterministic price is easily obtained with a composite curve.



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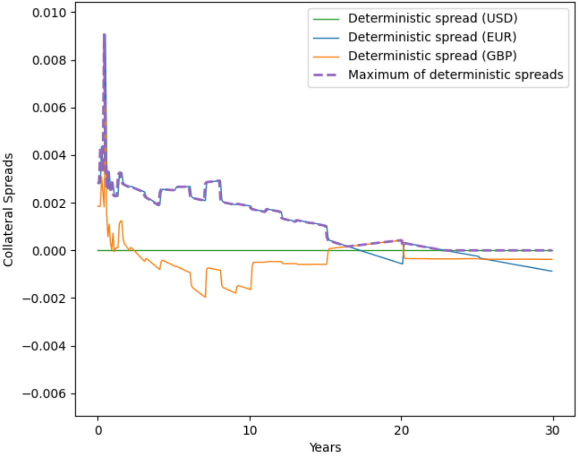
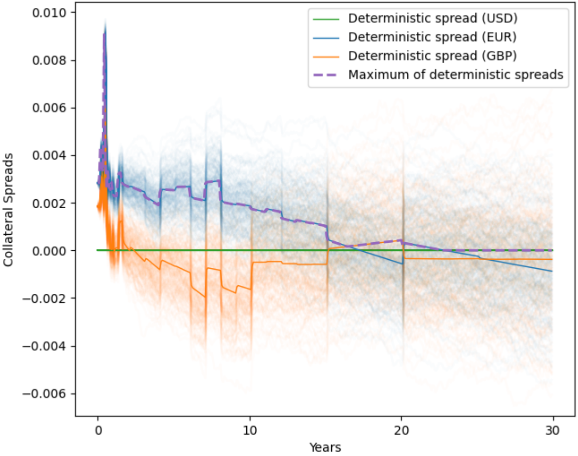


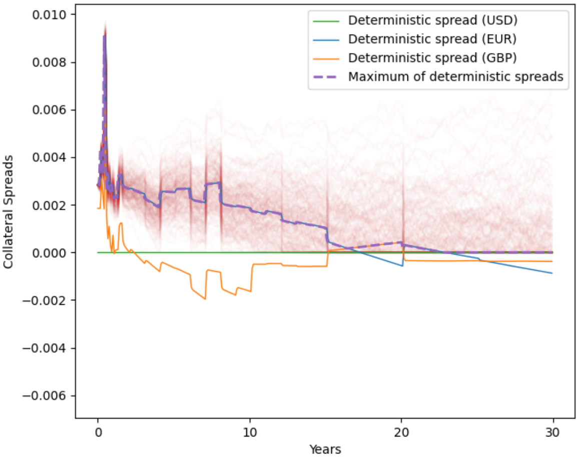
The model choice matters

With stochastic dynamics...



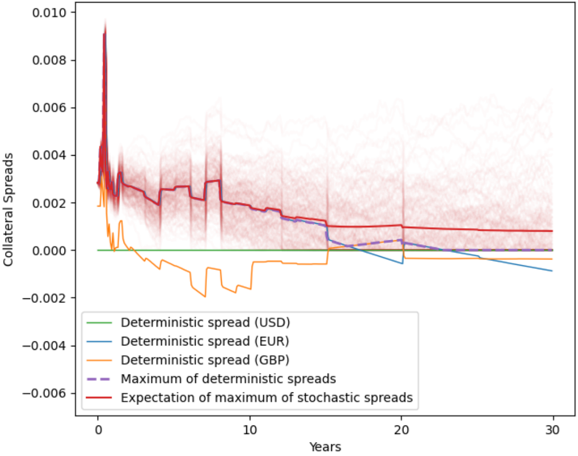
The model choice matters

With stochastic dynamics the maximum gets bigger!



The model choice matters

Jensen's inequality: $\max(0, \mathbb{E}[q_{\mathrm{EUR}}], \mathbb{E}[q_{\mathrm{GBP}}]) \leq \mathbb{E}[\max(0, q_{\mathrm{EUR}}, q_{\mathrm{GBP}})]$



How to construct a stochastic model

Spreads q_i behave similar to interest rates.
Modelling them with stochastic interest rate models (Gaussian processes) gives rise to the maximum process $M(t) = \max(0, q_1(t), \dots, q_N(t))$.

- No closed form marginal distributions
  $M(t) = \max(0, q_1(t), \dots, q_N(t))$.
- No closed form process distribution $\big(M(t)\big)_{t\in[0,T]}$.
- $\mathrm{CTD}(0, T) = \mathbb{E}\big[\exp(-\int_0^T M(t)\mathrm{d}t)\big] = \mathbb{E}\Big[\sum_{k=0}^{\infty} \frac{\left(-\int_0^T M(t)\mathrm{d}t\right)^k}{k!}\Big]$.

Stochastic model: Taylor approximation

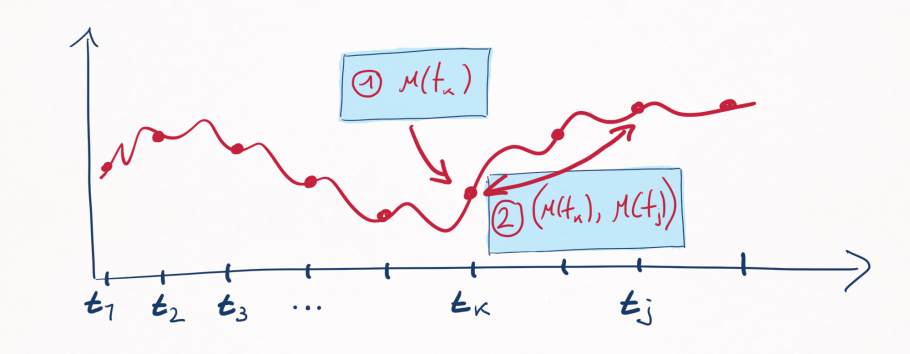
$$\mathrm{CTD}(0, T) = \mathbb{E}\Big[\sum_{k=0}^{\infty} \frac{\big(-\int_0^T M(t)\mathrm{d}t\big)^k}{k!}\Big].$$

First-order approximation is not precise enough, second-order approximation appears suitable:

$$\mathrm{CTD}(0, T) \approx \exp\Big(-\mathbb{E}\Big[\int_0^T M(t)\mathrm{d}t\Big]\Big)\Big(1 + \frac{1}{2}\mathbb{V}\mathrm{ar}\Big[\int_0^T M(t)\mathrm{d}t\Big]\Big),$$

but is still dependent on the marginal distributions $M(t)$ and covariances $\mathbb{C}\mathrm{ov}[M(t), M(s)]$!

Stochastic model: process approximation



1. Obtain marginal distribution from conditional independence assumption (common factor model)
2. Obtain process dynamics by approximation with similar processes for which it is known (Itô or mean-reverting diffusion)

Two models for CTD

Finally, we have three quantities:

- True value: $\mathrm{CTD}(0, T) = \mathbb{E}\big[\exp(-\int_0^T M(t)\mathrm{d}t\big]$.
- Deterministic model: $\mathrm{CTD}_{\mathrm{det}}(0, T)$.
- Stochastic model: $\mathrm{CTD}_{\mathrm{CF}}(0, T)$.

Stochastic and deterministic CTD models in practise

Consider an interest rate swap with the collateral choice option:

$$\begin{aligned} V^c(t) &= \sum_{k=1}^{m} \mathrm{CTD}(t, T_k)\, U_k(t) \\ &= \bar{N} \sum_{k=1}^{m} \mathrm{CTD}(t, T_k)\Big(\tau_k P(t, T_k)\big(\ell(t, T_{k-1}, T_k) - K\big)\Big). \end{aligned}$$

"True values" $\mathrm{CTD}(t, T_k)$ are unknown but we can approximate the discount factors with our models $\mathrm{CTD}_{\mathrm{det}}$ and $\mathrm{CTD}_{\mathrm{CF}}$.

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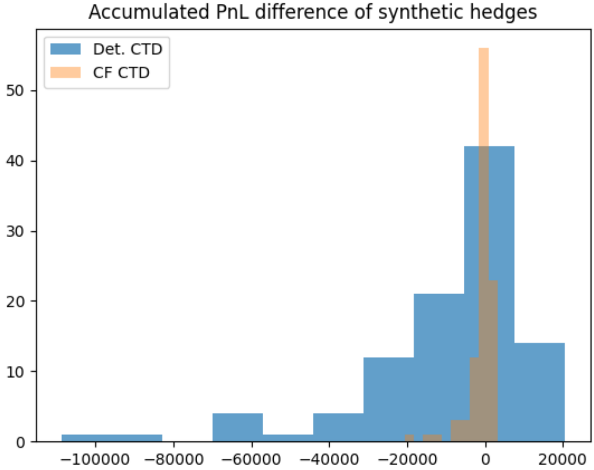
Consider portfolios Π_j, $j \in \{\mathrm{det}, \mathrm{CF}\}$:

$$\Pi_j(t) = V^c(t) - \bar{N} \sum_{T_k \geq t} \mathrm{CTD}_j(t, T_k) \tau_k (\ell(t, T_{k-1}, T_k) - K) P(t, T_k).$$

Accumulated error



Accumulated valuation error over the lifetime of a swap.

Hedging perspective

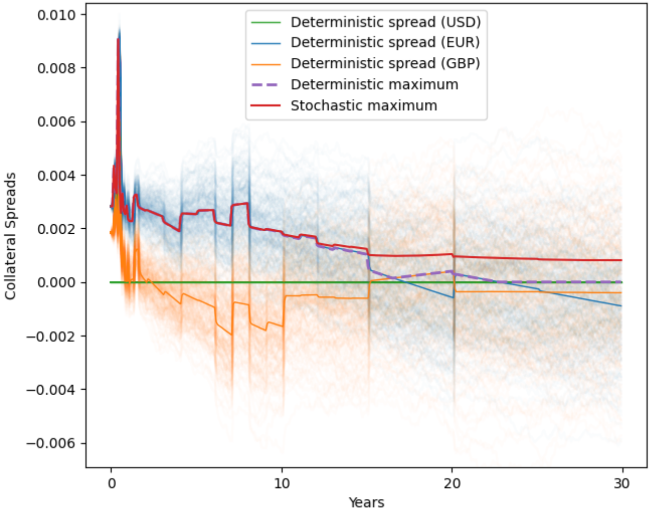
$$\Pi_j(t) = V^c(t) - \bar{N} \sum_{T_k \geq t} \mathrm{CTD}_j(t, T_k)\tau_k(\ell(t, T_{k-1}, T_k) - K)P(t, T_k).$$

These portfolios are comparable to a Delta hedge of the swap V^c.

$$V^c(t) = \sum_{k=1}^{m} \mathrm{CTD}(t, T_k)\, U_k(t).$$

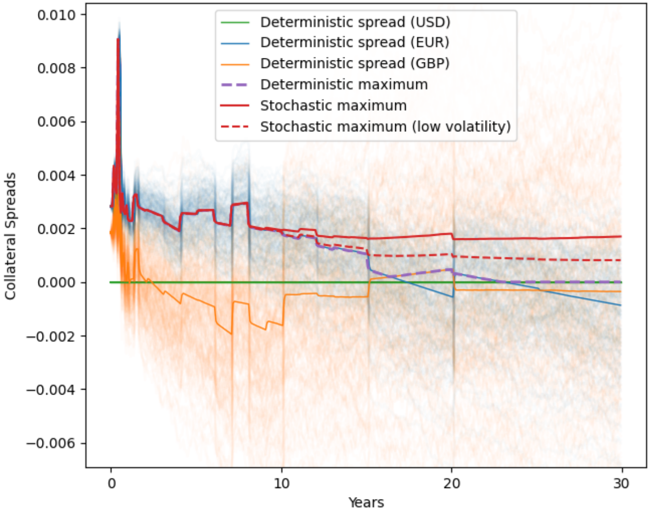
The linear product $\sum_k U_k$ only requires a Delta hedge, which risk factors are introduced by the CTD factor?

Hedging perspective



The stochastic maximum depends on all components at all times.

Hedging perspective



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What's in a hedge?

Risk factors are the "ingredients" of the asset price that change stochastically.

Changes in a risk factor X move the price of the asset:

$$X \uparrow \Longrightarrow \; V = f(X) \uparrow$$

Hedges are assets that move in opposite direction under these changes:

$$X \uparrow \Longrightarrow \; W = g(X) \downarrow$$

Together, asset and hedge are *neutral* to the risk factor:

$$X \uparrow \Longrightarrow \; (V + W) \uparrow\downarrow \text{(no change)}.$$

Call Option example (Black–Scholes market)

Consider Geometric Brownian motion stock

$$\mathrm{d}S_t = \mu(t) S_t \mathrm{d}t + \sigma(t) S_t \mathrm{d}W_t.$$

A European call option C pays $\max(S_T - K, 0)$, hence

$$S \uparrow \implies C \uparrow .$$

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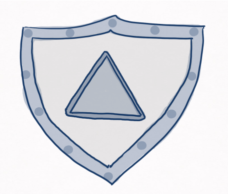

Delta-Hedge: Risk factor S has hedge $-S$:
$S\uparrow \implies -S\downarrow$

$S\uparrow \implies (C + \Delta S)\uparrow\downarrow$ is neutral (for some factor Δ).

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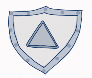
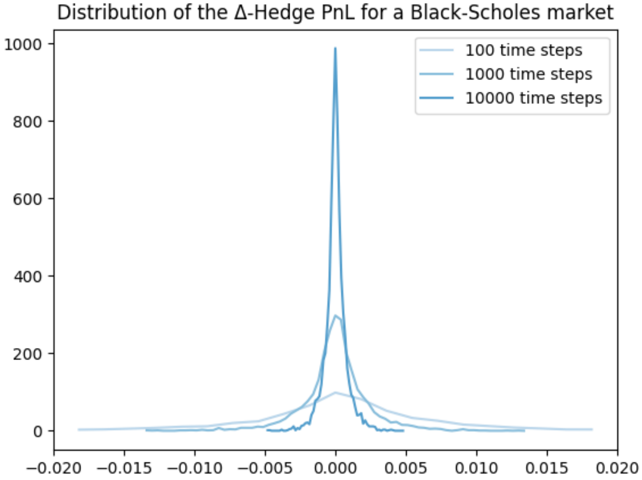


Delta-hedge portfolio $\Pi(t) = C_t + \Delta(t) S_t$



Call Option example (Heston market)

If stock S moves with Heston dynamics

$$\begin{aligned} \mathrm{d}S_t &= \mu S_t \mathrm{d}t + \sqrt{v_t} S_t \mathrm{d}W_t^S, \\ \mathrm{d}v_t &= \kappa(\theta(t) - v_t)\mathrm{d}t + \xi \sqrt{v_t} \mathrm{d}W_t^v, \end{aligned}$$

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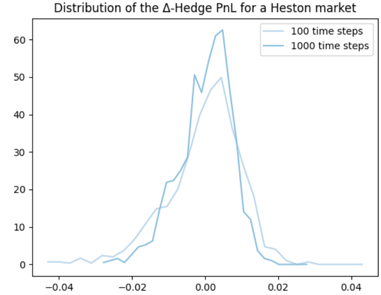

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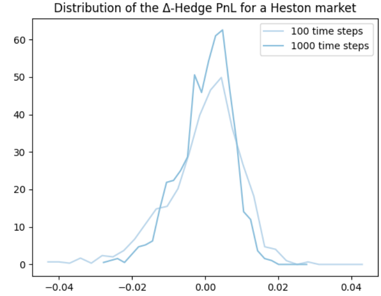
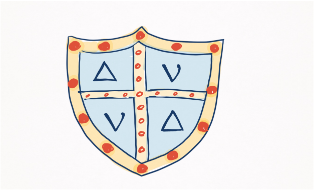


Volatility v is another risk factor!
$v \uparrow \implies C \uparrow$



Delta-Vega-hedge portfolio
$\Pi(t) = C_t + \Delta^*(t) S_t + \nu(t) C_t$

Hedging the collateral choice option

What are the risk factors in an asset with the collateral choice option:

$$V^c(t) = \mathrm{CTD}(t, T)V(t) \ ?$$

- V is linear $\implies$ only risk factor is its underlying $\implies$ Δ_V.
- CTD is convex $\implies$ $\Delta_{\mathrm{CTD}},\ \nu_{\mathrm{CTD}}$?

Collateral spreads cannot be traded, what is a hedging asset for them?

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Collateral spreads cannot be traded, what is a hedging asset for them?

1. Create artificial products "bond on collateral spreads"
   $Q_i(t, T) = \mathbb{E}[\exp(-\int_t^T q_i(s)\mathrm{d}s)]$
   and options on it.

2. Are there liquidly traded instruments that can serve as a proxy for the same risk factors?