Collateral valuation with a common factor

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Cheapest-To-Deliver (CTD) Collateral:

"What is the choice of the collateral currency worth?"

Fujii et al. ('10), Piterbarg et al. ('12, '13, '14), Sankovich et al. ('15)

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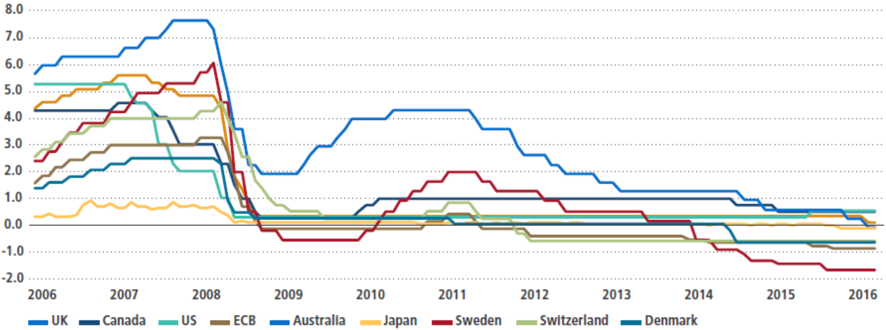
Central Bank Rates



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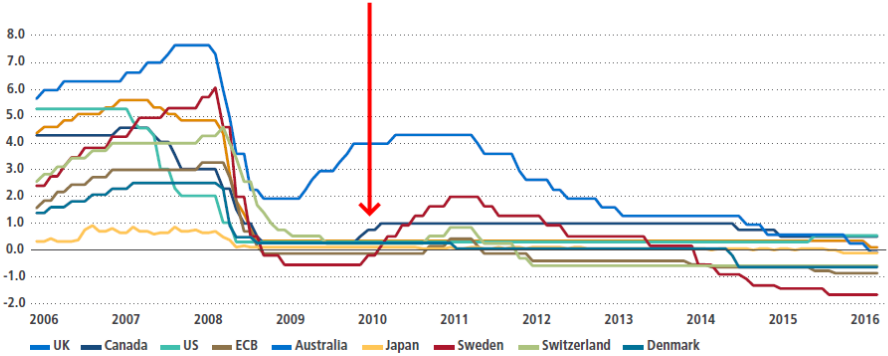
CTD literature emerges



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We consider the collateral choice option with free choice and substitution from a selection of rates.

One choice:

$$\{c\}$$

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T c(t)\mathrm{d}t} V(T)\right]$$

Many choices:

$$\{c_0, c_1, \ldots, c_N\}$$

$$\mathbb{E}^{\mathbb{Q}_0}\left[\mathrm{e}^{-\int_0^T \max(c_0(t), c_1(t), \ldots, c_N(t))\mathrm{d}t} V(T)\right]$$

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Each collateral rate creates its own measure: $\mathbb{Q}_0,\ \mathbb{Q}_1,\ \dots,\ \mathbb{Q}_N$.

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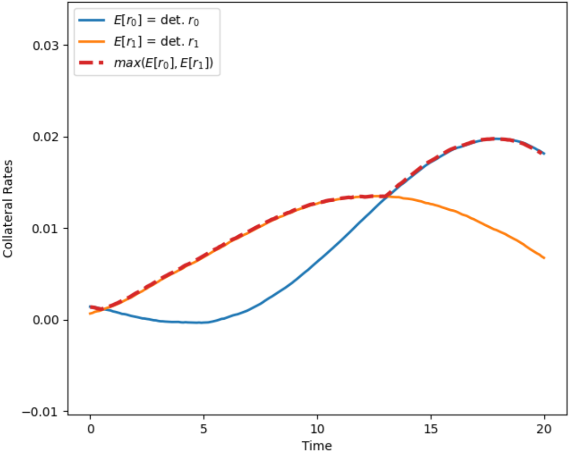
Many choices:

$$\{r_0, r_1, \ldots, r_N\}$$

$$\mathbb{E}^{\mathbb{Q}_0}\left[\mathrm{e}^{-\int_0^T \max(r_0(t), r_1(t), \ldots, r_N(t))\mathrm{d}t} V(T)\right]$$

Each collateral rate creates its own measure: $\mathbb{Q}_0$, $\mathbb{Q}_1$, $\ldots$, $\mathbb{Q}_N$.
Choose one ($\mathbb{Q}_0$) and translate all collateral rates to the corresponding currency.
"FX-adjusted collateral rates" $r_i = \mathrm{FX}_i^0(c_i)$

Stochastic modelling always leads to a bigger discount.



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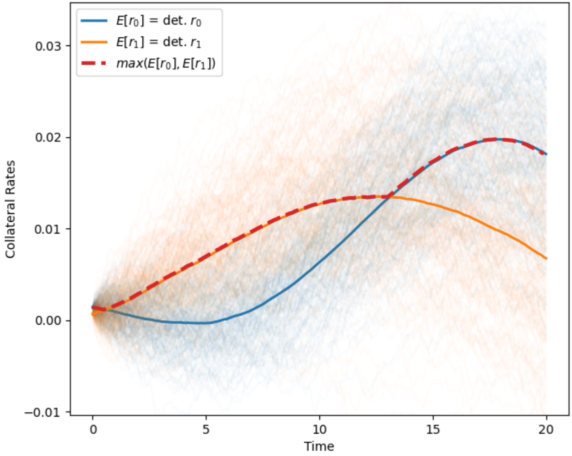

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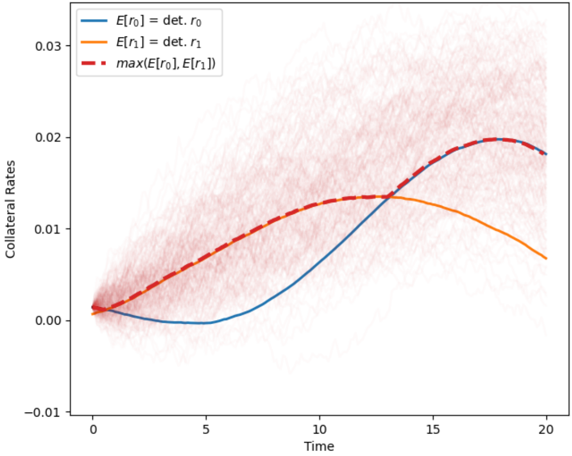

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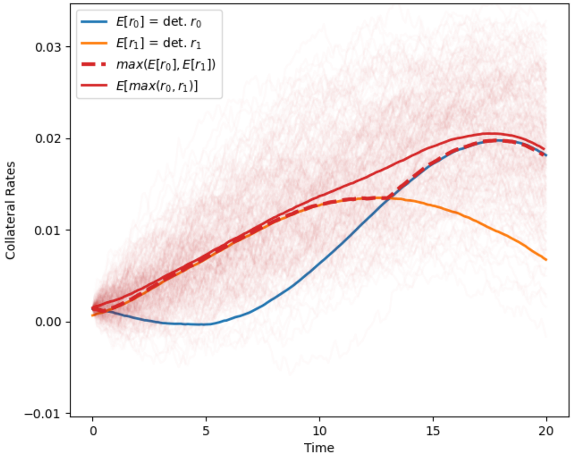

Jensen's inequality: $\max(\mathbb{E}^{\mathbb{Q}_0}[r_0(t)], \mathbb{E}^{\mathbb{Q}_0}[r_1(t)]) \leq \mathbb{E}^{\mathbb{Q}_0}[\max(r_0(t), r_1(t))]$

Instead of FX-adjusted collateral rates, we use collateral spreads:

$$\max\bigl(r_0(t), r_1(t), \dots, r_N(t)\bigr) = r_0(t) + \max\bigl(0, (r_1 - r_0)(t), \dots, (r_N - r_0)(t)\bigr)$$

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}_0}\left[e^{-\int_0^T \max\left(r_0(t), r_1(t), \dots, r_N(t)\right) \mathrm{d}t}\right] \\
&= \underbrace{\mathbb{E}^{\mathbb{Q}_0}\left[e^{-\int_0^T r_0(t) \mathrm{d}t}\right]}_{} \; \mathbb{E}^{T}\left[e^{-\int_0^T \max\left(0, (r_1 - r_0)(t), \dots, (r_N - r_0)(t)\right) \mathrm{d}t}\right] \\
&= \qquad P_0(0, T) \qquad \mathbb{E}^{T}\left[e^{-\int_0^T \max\left(0, q_1(t), \dots, q_N(t)\right) \mathrm{d}t}\right]
\end{aligned}$$

Collateral spreads have much less volatility and their maximum is well described by a second order Taylor expansion.

The spreads q_i are modelled as correlated Hull-White processes under the T-forward measure:

$$\mathrm{d}q_i(t) = \kappa_i\big(\theta_i(t) - q_i(t)\big)\mathrm{d}t + \xi_i \mathrm{d}W_i(t)$$

Finite Dimensional Distribution:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ q_1(t_1) & q_1(t_2) & q_1(t_3) & q_1(t_4) & q_1(t_5) & q_1(t_6) \\ q_2(t_1) & q_2(t_2) & q_2(t_3) & q_2(t_4) & q_2(t_5) & q_2(t_6) \\ q_3(t_1) & q_3(t_2) & q_3(t_3) & q_3(t_4) & q_3(t_5) & q_3(t_6) \end{pmatrix}$$

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Define maximum process

$$M(t) = \max\big(0, q_1(t), \ldots, q_N(t)\big)$$

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1) The marginal distribution $\max(0, q_1(t), \ldots, q_N(t))$ is not tractable.

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We need it all!

$$\mathbb{E}^T\Big[\exp\big(-\int_0^T M(t)\mathrm{d}t\big)\Big]$$

The problem can be simpified by order reduction.

For collateral spreads, second order approximation is sufficient.

$$\begin{aligned} &\mathbb{E}^{T}\Big[\exp\big(-\int_0^T M(t)\mathrm{d}t\big)\Big] \\ &\approx \exp\Big(\mathbb{E}^{T}\big[-\int_0^T M(t)\mathrm{d}t\big]\Big)\Big(1+\frac{1}{2}\mathrm{Var}\big[\int_0^T M(t)\mathrm{d}t\big]\Big) \end{aligned}$$

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&= \exp\Big(-\int_0^T \mathbb{E}^T\big[M(t)\big]\mathrm{d}t\Big)\Big(1+\frac{1}{2}\mathrm{Var}\big[\int_0^T M(t)\mathrm{d}t\big]\Big)
\end{aligned}
$$

First order term depends only on the marginal distribution.

At each time t, we model the spreads $q_i(t)$ conditionally independent.

Common factor approximation:

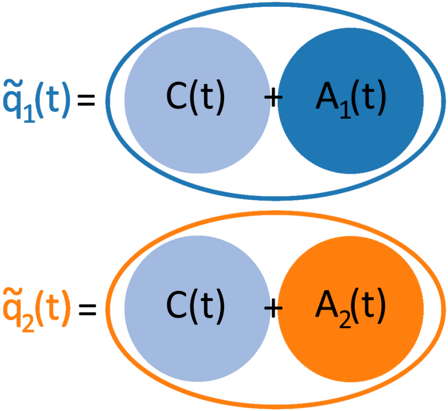
$\widetilde{q}_i(t) = C(t) + A_i(t)$

with independent
normal distributions

$C(t)$ and $A_i(t)$

such that

$\widetilde{q}_i(t) \sim q_i(t)$.



The common factor maximum can be expressed in terms of independent random variables.

$$\begin{aligned}\widetilde{M}(t) &= \max\bigl(0, \widetilde{q}_1(t), \ldots, \widetilde{q}_N(t)\bigr) \\ &= \max\bigl(0, C(t) + A_1(t), \ldots, C(t) + A_N(t)\bigr) \\ &= \max\bigl(0, C(t) + \max\bigl(A_1(t), \ldots, A_N(t)\bigr)\bigr)\end{aligned}$$

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This makes the CDF of $\widetilde{M}(t)$ available.

The CDF of $\widetilde{M}(t) = \max\big(0, C(t) + \max(A_1(t), \ldots, A_N(t))\big)$ is

$$\mathbb{P}^T[\widetilde{M}(t) \leq x] = \begin{cases} 0, & x \leq 0, \\ \mathbb{P}^T[C(t) + \max(A_1(t), \ldots, A_N(t)) \leq x], & x > 0. \end{cases}$$

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The CDF of the maximum is

$$F_{\max(A_i(t))}(x) = \mathbb{P}^T[\max(A_1(t), \dots, A_N(t)) \le x] = \prod_{i=1}^{N} \mathbb{P}^T[A_i(t) \le x].$$

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The CDF of the sum of independent random variables is

$$\mathbb{P}^T[C(t) + \max(A_1(t), \ldots, A_N(t)) \le x] = \big(f_{C(t)} * F_{\max(A_i(t))}\big)(x).$$

density of $C(t)$ Convolution

With the CDF of $\widetilde{M}(t)$, any moment can be computed.

$$\mathbb{E}^{T}\left[\widetilde{M}(t)^{\ell}\right] = \int_{0}^{\infty} \ell x^{\ell-1}\Big(1 - \big(f_{C(t)} * F_{\max(A_i(t))}\big)(x)\Big)\mathrm{d}x$$

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Recall the second order approximation goal:

$$\exp\Big(\int_0^T \mathbb{E}^T\left[M(t)\right]\mathrm{d}t\Big)\Big(1 + \frac{1}{2}\mathrm{Var}\big[\int_0^T M(t)\mathrm{d}t\big]\Big)$$

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The variance of the integral is harder to estimate.

$$\mathrm{Var}\Big[\int_0^T M(t)\mathrm{d}t\Big] = \mathbb{E}^T\Big[\int_0^T \int_0^T M(t)M(s)\mathrm{d}t\mathrm{d}s\Big] - \mathbb{E}^T\Big[\int_0^T M(t)\mathrm{d}t\Big]^2$$

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1) Ito Process $\mathrm{d}X(t) = g(t)\mathrm{d}t + h(t)\mathrm{d}B(t)$

Choose coefficients g, h such that

$\mathbb{E}[X(t)] = \mathbb{E}[\widetilde{M}(t)]$, $\mathrm{Var}[X(t)] = \mathrm{Var}[\widetilde{M}(t)]$.

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This variance estimator neglects mean-reversion but is quickly computed.

2) Mean-reverting estimator

Based on Sankovich & Zhu '15

Heuristically:

$$\mathrm{Var}\Big[\int_0^T M(t)\mathrm{d}t\Big] \approx \mathrm{Var}\Big[\int_0^T Z(t)\mathrm{d}t\Big]$$

with a mean-reverting process $Z(t)$ such that

$$\mathrm{Var}[Z(t)] = \mathrm{Var}[\widetilde{M}(t)]$$

and the mean-reversion speed is

$$\widetilde{\kappa}(t) = \sum_{i=1}^{N} \kappa_i \mathbb{P}^T\big[\widetilde{q}_i(t) = \widetilde{M}(t)\big].$$

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available analytically

Are there no downsides to the common factor approximation?

Analytical tractability comes at cost of simplified correlation structure.

$$\mathrm{Cov}[\widetilde{q}_i(t), \widetilde{q}_j(t)] = \mathrm{Cov}[C(t) + A_i(t), C(t) + A_j(t)] = \mathrm{Var}[C(t)]$$

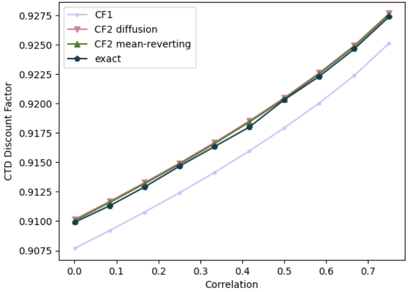
Thus

$$\mathrm{corr}[\widetilde{q}_i(t), \widetilde{q}_j(t)] = \frac{\mathrm{Var}[C(t)]}{\sqrt{\mathrm{Var}[q_i(t)]\mathrm{Var}[q_j(t)]}} \geq 0$$

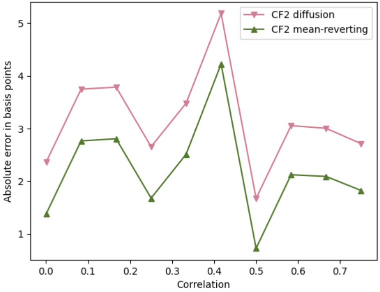
This has only one degree of freedom, so for more than three currencies $(q_0 = 0, q_1, q_2)$ the correlation structure must be approximated.

Three currencies: Correlation

Effect on Discount Factor

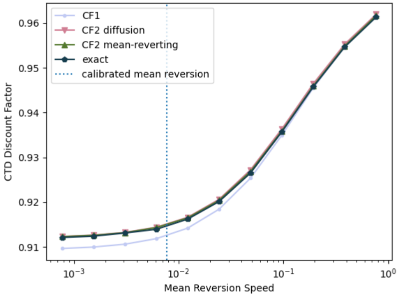


Error of Common Factor Model



Three currencies: Mean-reversion speed

Effect on Discount Factor



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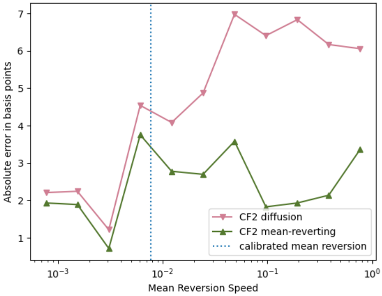

Conclusions

- Semi-analytical result
  with restriction to correlation range (or complexity)
- Fast computation even for large number of currencies
- Model error well within range of parameter error