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Unified model for XVA including Interest Rates and Rating

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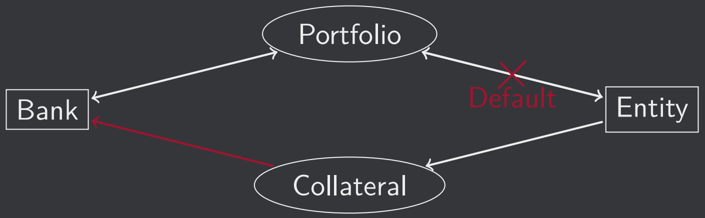
Setting of the thesis



Collateral reduces the potential loss at the default and reduces the value of XVA making the financial derivative more attractive to customers but can increase the default probability!

Aim of the thesis

We would like to minimize the collateral-inclusive CVA

$$\min_{\mathrm{C}\in ?} \mathbb{E}^{\mathbb{Q}}\left[\mathrm{LGD}\,\exp\left(-\int_t^{\tau} r_s ds\right)\mathbb{1}_{\tau<T}\left(\mathrm{V}_{\tau}^{+}-\mathrm{C}_{\tau}^{+}\right)^{+}\middle|\,\mathcal{G}_t\right]. \qquad \text{(CVA)}$$

1. The loss-given-default (LGD) will be constant and is equal to 0.6;
2. The time of default prior to the end of contracts $T > 0$ of an entity is denoted by τ;
3. The portfolio at time t between the counterparty and an entity is denoted by V_t;
4. The collateral account at time t by C_t;
5. The discount factor seen from time t up to time u by $\exp\left(-\int_t^u r_s ds\right)$.

Aim of the thesis

$$\min_{C \in ?} \mathbb{E}^{\mathbb{Q}} \left[\mathrm{LGD} \exp\left(- \int_t^{\tau} r_s ds \right) \mathbb{1}_{\tau < T} \left(V_{\tau}^{+} - C_{\tau}^{+} \right)^{+} \Big| \mathcal{G}_t \right]. \tag{CVA}$$

1. The unconstraint problem is easily solved by $C_t = V_t$, this is called *perfect collateralization*;
2. Posting collateral is expensive and counterparties would like to avoid it;
3. Therefore the aim is to minimize (CVA) under the constraint of as little collateral postings as possible;
4. One way to do this, is to take the creditworthiness of a counterparty into account, which we will see in Section 2.

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Table of Contents

Di Francesco, Marco and Kevin Kamm (2021). "How to handle negative interest rates in a CIR framework". In: *SeMA Journal*. ISSN: 2281-7875. URL: https://doi.org/10.1007/s40324-021-00267-w.

Di Francesco, Marco and Kevin Kamm (2022). "On the Deterministic-Shift Extended CIR Model in a Negative Interest Rate Framework". In: *International Journal of Financial Studies* 10(2). ISSN: 2227-7072. URL: https://www.mdpi.com/2227-7072/10/2/38.

Kamm, Kevin (2022). *An introduction to rating triggers for collateral-inclusive XVA in an ICTMC framework*. URL: https://arxiv.org/abs/2207.03883.

Kamm, Kevin and Michelle Muniz (2022). *A novel approach to rating transition modelling via Machine Learning and SDEs on Lie groups*. URL: https://arxiv.org/abs/2205.15699.

Kamm, Kevin, Stefano Pagliarani and Andrea Pascucci (2021). "On the Stochastic Magnus Expansion and Its Application to SPDEs". In: *Journal of Scientific Computing* 89(3), p. 56. ISSN: 1573-7691. URL: https://doi.org/10.1007/s10915-021-01633-6.

Kamm, Kevin, Stefano Pagliarani and Andrea Pascucci (2022). *Numerical solution of kinetic SPDEs via stochastic Magnus expansion*. URL: https://arxiv.org/abs/2207.09776.

Interest rates

Calibration to swaptions

$$\min_{C\in ?} \mathbb{E}^{\mathbb{Q}}\left[\mathrm{LGD}\exp\left(-\int_t^{\tau} r_s ds\right)\mathbb{1}_{\tau<T}\left(\mathrm{V}_{\tau}^{+}-\mathrm{C}_{\tau}^{+}\right)^{+}\middle|\mathcal{G}_t\right]$$

joint work with Marco Di Francesco

Idea

We want to study negative interest rates in a Cox-Ingersoll-Ross framework. In particular, we set

$$r(t) = x(t) - y(t) + \psi(t),$$

where for $z \in \{x, y\}$

$$dz(t) = k_z(\theta_z - z(t))dt + \sigma_z\sqrt{z(t)}dW_z(t), \quad z(0) = z_0 \qquad \text{(CIR)}$$

are independent and $\psi(t)$ is the deterministic shift extension

$$\psi(t) := \mathrm{f}^M(0, t) - \mathrm{f}(0, t)$$

with $\mathrm{f}^M(0, t)$, $\mathrm{f}(0, t)$ the market, model instantaneous forward rate, respectively.

Swaps

The net value of a $T_0 \times (T_N - T_0)$ payer and receiver swap at time $t \leq T_0$ is given by

$$\mathrm{Swap}(t; K, \zeta) := \zeta \left(\mathrm{P}(t, T_0) - \mathrm{P}(t, T_N) - K \sum_{i=1}^{N} \alpha_i \mathrm{P}(t, T_i) \right) \tag{1.1}$$

where $\alpha_i = T_i - T_{i-1}$ is the day-count convention and K the fixed rate,

Swaptions

Let us first of all make the following observation: The payer ($\zeta = 1$) and receiver ($\zeta = -1$) swap value (1.1) can both be rewritten as

$$\mathrm{Swap}(t; K, \zeta) := \sum_{i=0}^{N} a_i^{\zeta} \mathrm{P}(t, T_i),$$

where a_i^{ζ} are equal to

$$a_0^{\zeta} := \zeta, \qquad a_N^{\zeta} := -\zeta\left(1 + K\alpha_N\right), \qquad a_i^{\zeta} := -\zeta K\alpha_i, \quad i = 1, \dots, N-1.$$

Now, with this notation, we can write the swaption prices under the forward measure as

$$\begin{aligned}\mathrm{Swaption}(t; K, \zeta) &= \mathrm{P}(t, T_0)\mathbb{E}^{\mathbb{Q}^{T_0}}\left[\left(\mathrm{Swap}(T_0; K, \zeta)\right)^+ \middle| \mathcal{F}_t\right] \\ &\stackrel{!}{=} \mathrm{P}(t, T_0)\int_0^{\infty} x f(x) dx,\end{aligned}$$

for an unknown density function f.

Gram-Charlier expansion

Assume that a random variable Y has the continuous density function f and has finite cumulants c_k, $k \geq 1$. Then the following holds:
f can be expanded as

$$f(x) = \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n\left(\frac{x - c_1}{\sqrt{c_2}}\right) \varphi\left(\frac{x - c_1}{\sqrt{c_2}}\right),$$

where H_n are the probabilist's Hermite polynomials and φ the probability density function of the standard normal distribution, as well as $q_0 = 1$, $q_1 = q_2 = 0$, and for $n \geq 3$

$$q_n = \frac{1}{n!}\mathbb{E}\left[H_n\left(\frac{Y - c_1}{\sqrt{c_2}}\right)\right] = \sum_{m=1}^{\lfloor \frac{n}{3} \rfloor} \sum_{\substack{k_1 + \cdots + k_m = n \\ k_i \geq 3}} \frac{c_{k_1} \cdots c_{k_m}}{m! k_1! \cdots k_m!} \left(\frac{1}{\sqrt{c_2}}\right)^n.$$

Gram-Charlier expansion

In our case, we have for any $a \in \mathbb{R}$

$$\mathbb{E}\left[Y \mathbb{1}_{Y \geq a}\right] = c_1 \mathcal{N}\left(\frac{c_1 - a}{\sqrt{c_2}}\right) + \sqrt{c_2}\varphi\left(\frac{c_1 - a}{\sqrt{c_2}}\right) + \sum_{n=3}^{\infty} (-1)^{n-1} q_n \varphi\left(\frac{c_1 - a}{\sqrt{c_2}}\right)\left[a H_{n-1}\left(\frac{c_1 - a}{\sqrt{c_2}}\right) - \sqrt{c_2} H_{n-2}\left(\frac{c_1 - a}{\sqrt{c_2}}\right)\right],$$

where furthermore $\mathcal{N}$ denotes the cumulative distribution function of the standard normal distribution.

In particular, we have

$$q_3 = \frac{c_3}{3! c_2^{\frac{3}{2}}}, \quad q_4 = \frac{c_4}{4! c_2^{\frac{4}{2}}}, \quad q_5 = \frac{c_5}{5! c_2^{\frac{5}{2}}}, \quad q_6 = \frac{c_6 + 10 c_3^2}{6! c_2^{\frac{6}{2}}}, \quad q_7 = \frac{c_7 + 35 c_3 c_4}{7! c_2^{\frac{7}{2}}}.$$

Gram-Charlier expansion

Therefore, it remains to find the cumulants c_i, usually 7 are enough. For this, one proceeds as follows:

1. Use the fact that cumulants and moments are one-to-one;
2. Derive the bond and swap moments;
3. For this, Riccati equations have to be solved;
4. Truncate the Gram-Charlier expansion and use it for approximating swaption prices.

Ratings

Historical and Market Data
A Lie Group perspective
The stochastic Langevin equation

$$\min_{C \in ?} \mathbb{E}^{\mathbb{Q}} \left[\mathrm{LGD} \exp\left(- \int_t^{\tau} r_s ds \right) \mathbb{1}_{\tau < T} \left(\mathrm{V}_{\tau}^{+} - \mathrm{C}_{\tau}^{+} \right)^{+} \middle| \mathcal{G}_t \right]$$

joint work with Michelle Muniz and Luca Caputo

Ratings

Ratings are an indicator of creditworthiness and usually denoted by

$$\text{best ratings} \quad \mathbf{A} > \mathbf{B} > \mathbf{C} > \mathbf{D} \quad \text{worst rating}$$

The rating **D** denotes the default or bankruptcy of an entity. We will assume that an entity cannot recover from default.

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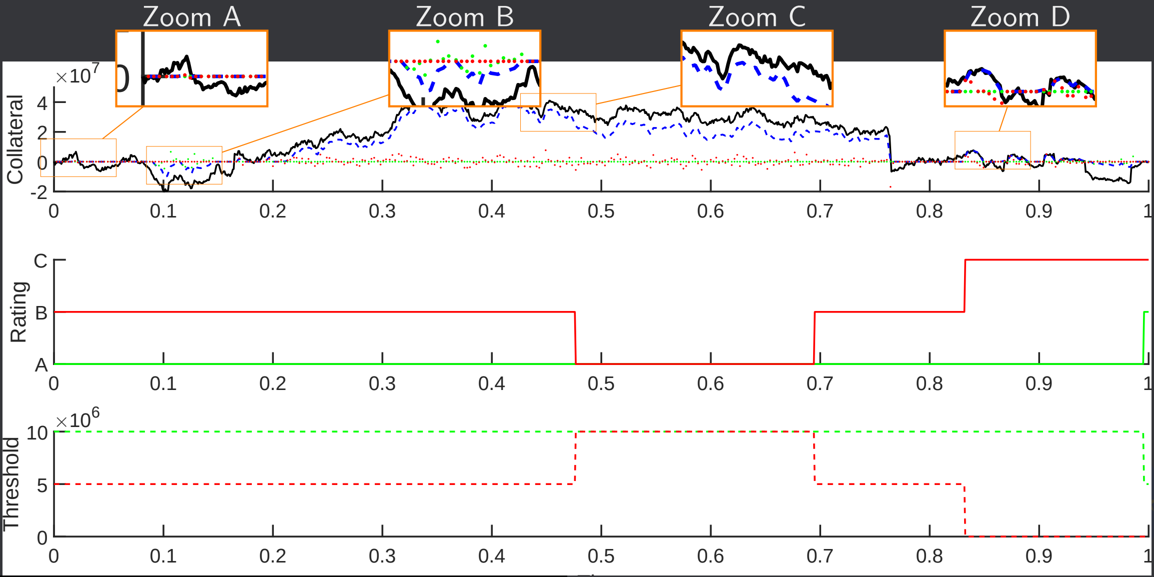
With this concept, we would like to define the collateral account as

$$\mathrm{C}_t := f\left(V_t, X_t\right),$$

where $X_t := \left(X_t^B, X_t^C\right)$ is a stochastic process whose values are the rating of a bank and a counterparty at time t.


Zoom A
Zoom B
Zoom C
Zoom D
×10^7
Collateral
4
2
0
-2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Rating
C
B
A
×10^6
Threshold
10
5
0

Rating matrices

Example of a one year rating matrix under the historical measure:

From \ To	A	B	C	D
A	0.9395	0.0566	0.0037	2.7804e-04
B	0.0092	0.9680	0.0211	0.0017
C	6.2064e-04	0.0440	0.8154	0.1400
D	0	0	0	1

1. Probability of transitioning from **B** to **C** in one year is 2.11 %
2. Absorbing default state
3. Rows sum up to one
4. Under the risk-neutral measure only the default column is known from Credit-Default-Swaps (CDS) with usually slightly higher probabilities

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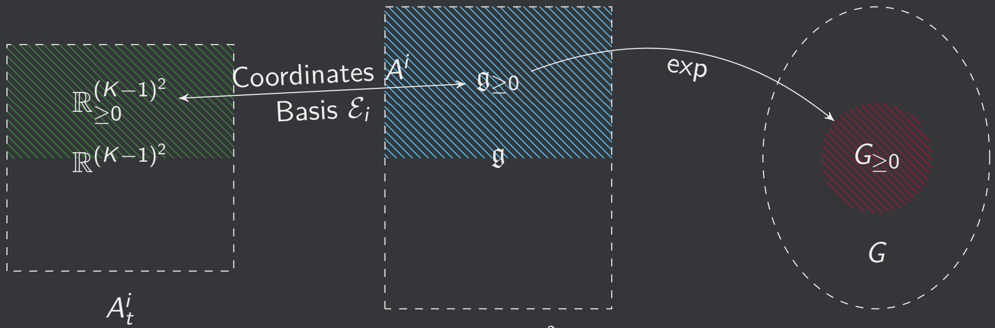
Stochastic Matrices form a Lie-Group



$A_t := \sum_{i=1}^{(K-1)^2} A^i_t \mathcal{E}_i$

gEM: $R_{t_{k+1}} = R_{t_k} \exp(A_{t_{k+1}})$

A model in the Lie-Algebra

We need to ensure that the gEM has values not only in the Lie-Group G but in the subspace of stochastic matrices $G_{\geq 0}$. One sufficient condition is to ensure monotonically increasing paths in the Lie-Algebra.
Therefore, we define our model in the Lie-Algebra under the historical measure by

$$\begin{aligned} dA_t^i &= \left| Y_t^i \right|^{a_i} dt \\ dY_t^i &= b_i dt + \sigma_i dW_t^i, \quad Y_0^i = 0. \end{aligned} \tag{Langevin}$$

We can derive the dynamics under a risk-neutral measure by applying the usual Girsanov theorem to Y_t^i.
Also notice that (Langevin) has Langevin-like dynamics, which we will come back to later.

Calibration

1. Under the historical measure, we use a Deep-Neural-Network called TimeGAN to analyse the distribution of historical time-series data and match the moments of our model and TimeGAN data;
2. Under the risk-neutral measure, we calibrate the change of measure parameters, such that the model has close probabilities of default compared to the market data;
3. A rating process can now be simulated with a nested Stochastic Simulation Algorithm (SSA) leading to a doubly stochastic process X_t^B and X_t^C for the bank and counterparty.

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Collateral-inclusive bilateral XVA

XVA with the different collateral agreements (no, perfectly and rating triggers) using $\mathrm{LGD}_B = 0.6$, as well as $\mathrm{LGD}_C = 0.6$ with $M = 10000$ simulations and thresholds defined as before.

XVA	Uncollateralized	Rating Triggers	Perfectly collateralized
DVA	1015922	587335	351276
CVA	896413	376938	271492

A filtering problem

For the available data we have an information mismatch under the historical and risk-neutral measure.

	Historical data	Risk neutral data
Entity	(unobserved)	observed
Sector	observed	(unobserved)

At the moment we are studying the stochastic Langevin equation for this problem, which emerges if one applies the Fokker-Planck equation to a special case of (Langevin). This leads to an SPDE with two spatial dimensions, for which we found an efficient numerical scheme based on the Magnus expansion.

Magnus expansion

Heuristical derivation
Expansion formulas
SPDE

joint work with Stefano Pagliarani and Andrea Pascucci

Idea

Solve the matrix-valued SDE

$$dX_t = BX_t dt + AX_t dW_t, \quad X_0 = I_d$$

by assuming that there exists a solution $X_t = \exp(Y_t)$ for small times $t > 0$ depending on a stopping time and

$$Y_t = \int_0^t \mu(Y_s)\, ds + \int_0^t \sigma(Y_s)\, dW_s, \quad Y_0 = 0_{\mathbb{R}^{d \times d}}.$$

Determine μ and σ

$$
\begin{aligned}
dX_t &= BX_t + AX_t dW_t && \leftarrow \text{Equation}\\
&= B \exp(Y_t) + A \exp(Y_t)\, dW_t && \leftarrow \text{Assumption}\\
&= d \exp(Y_t) && \leftarrow \text{Assumption}\\
&= \left(\mathcal{L}_{Y_t}(\mu(Y_s)) + \frac{1}{2} \mathcal{Q}_{Y_t}(\sigma(Y_t), \sigma(Y_t)) \right) \exp(Y_t)\, dt && \leftarrow \text{Itô's formula}\\
&\quad + \mathcal{L}_{Y_t}(\sigma(Y_t)) \exp(Y_t)\, dW_t .
\end{aligned}
$$

A comparison of coefficients yields

$$
\begin{aligned}
B &\stackrel{!}{=} \mathcal{L}_{Y_t}(\mu(Y_t)) + \frac{1}{2} \mathcal{Q}_{Y_t}(\sigma(Y_t), \sigma(t, Y_t))\\
A &\stackrel{!}{=} \mathcal{L}_{Y_t}(\sigma(Y_t)) .
\end{aligned}
$$

Determine μ and σ

Inverting $\mathcal{L}_Y$ by using Baker's lemma yields

$$\sigma\left(Y_t\right) \equiv \sum_{n=0}^{\infty} \frac{\beta_n}{n!} \mathrm{ad}_{Y_t}^{n}\left(A\right) \tag{3.2}$$

$$\mu\left(Y_t\right) \equiv \sum_{k=0}^{\infty} \frac{\beta_k}{k!} \mathrm{ad}_{Y_t}^{k}\left(B - \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\mathrm{ad}_{Y_t}^{n}\left(\sigma\left(Y_t\right)\right)}{(n+1)!}\frac{\mathrm{ad}_{Y_t}^{m}\left(\sigma\left(Y_t\right)\right)}{(m+1)!} + \frac{\left[\mathrm{ad}_{Y_t}^{n}\left(\sigma\left(Y_t\right)\right), \mathrm{ad}_{Y_t}^{m}\left(\sigma\left(Y_t\right)\right)\right]}{(n+m+2)(n+1)!m!}\right) \tag{3.3}$$

where β_n denote the Bernoulli numbers, e.g. $\beta_0 = 1$, $\beta_1 = -\frac{1}{2}$, $\beta_2 = \frac{1}{6}$, $\beta_3 = 0$ and $\beta_4 = -\frac{1}{30}$.

Solve the SDE by Picard-iteration

Now, we solve the SDE for Y_t by Picard-iteration

$$Y_t^n = \int_0^t \mu\left(Y_s^{n-1}\right) ds + \int_0^t \sigma\left(Y_s^{n-1}\right) dW_s. \tag{3.4}$$

In order to derive the Magnus expansion formulas, we will introduce some bookkeeping parameters $\epsilon, \delta > 0$ and substitute A by ϵA, as well as B by δB.
The Magnus expansion of order

1. one will contain all the terms of Y_t^1 with ϵ^1 and δ^1;
2. two will contain all the terms of Y_t^2 with ϵ^2, δ^2 and $\epsilon^1\delta^1$ plus all the terms of Y_t^1;
3. three will contain all the terms of Y_t^3 with ϵ^3, δ^3, $\epsilon^2\delta^1$,$\epsilon^1\delta^2$ plus all the terms of Y_t^2;
4. ...

Order 1, 2, 3

1. order 1

$$Y_t^1 = \int_0^t B ds + \int_0^t A dW_s = Bt + AW_t.$$

2. order 2

$$\begin{aligned} Y_t^2 &= Bt - \frac{1}{2}A^2 t + \frac{1}{2}[B,A]\int_0^t W_s ds + AW_t - \frac{1}{2}[B,A]\left(tW_t - \int_0^t W_s ds\right) \\ &= Y_t^1 - \frac{1}{2}A^2 t + [B,A]\int_0^t W_s ds - \frac{1}{2}[B,A]\,tW_t. \end{aligned}$$

3. order 3

$$\begin{aligned} Y_t^3 = Y_t^2 &+ [[B,A],A]\left(\frac{1}{2}\int_0^t W_s^2 ds - \frac{1}{2}W_t\int_0^t W_s ds + \frac{1}{12}tW_t^2\right) \\ &+ [[B,A],B]\left(\int_0^t sW_s ds - \frac{1}{2}t\int_0^t W_s ds - \frac{1}{12}t^2 W_t\right). \end{aligned}$$

General parabolic SPDE

We want to discretize the following SPDE in space only to apply the Magnus expansion

$$
\begin{cases} du_t(x,v) = \Big(h(x,v)u_t(x,v) + f^x(x,v)\partial_x u_t(x,v) + f^v(x,v)\partial_v u_t(x,v) \\ \qquad + \frac{1}{2}g^{xx}(x,v)\partial_{xx}u_t(x,v) + g^{xv}(x,v)\partial_{xv}u_t(x,v) + \frac{1}{2}g^{vv}(x,v)\partial_{vv}u_t(x,v)\Big)dt \\ \qquad + \left(\sigma(x,v)u_t(x,v) + \sigma^x(x,v)\partial_x u_t(x,v) + \sigma^v(x,v)\partial_v u_t(x,v)\right)dW_t \\ u_0(x,v) = \phi(x,v). \end{cases} \tag{SPDE}
$$

Finite Differences

Let $\mathbb{X}^{n_x}_{a_x,b_x}$ be the grid for the position of a particle with $n_x + 2$ points on the subset $[a_x, b_x] \subset \mathbb{R}$ and $\mathbb{V}^{n_v}_{a_v,b_v}$ be the grid of its velocity with $n_v + 2$ points on the subset $[a_v, b_v] \subset \mathbb{R}$

$$\mathbb{X}^{n_x}_{a_x,b_x} := \{x_i^{n_x} \in [a_v, b_v] : x_i^{n_v} = a_x + i\Delta x,\ i = 0, \dots, n_v + 1\}, \qquad \Delta x := \frac{b_x - a_x}{n_x + 1},$$
$$\mathbb{V}^{n_v}_{a_v,b_v} := \left\{v_j^{n_v} \in [a_v, b_v] : v_j^{n_v} = a_v + j\Delta v,\ j = 0, \dots, n_v + 1\right\}, \qquad \Delta v := \frac{b_v - a_v}{n_v + 1},$$

For simplicity we set $d = n_x = n_v$, $[a_x, b_x] = [a_v, b_v] = [-4, 4]$ during our experiments later on.

Finite Differences

We will impose zero-boundary conditions and therefore define the central finite-difference matrices

$$D^x := \frac{1}{2\Delta x}\mathrm{tridiag}^{n_x,n_x}(-1,0,1), \qquad D^v := \frac{1}{2\Delta v}\mathrm{tridiag}^{n_v,n_v}(-1,0,1),$$
$$D^{xx} := \frac{1}{(\Delta x)^2}\mathrm{tridiag}^{n_x,n_x}(1,-2,1), \qquad D^{vv} := \frac{1}{(\Delta v)^2}\mathrm{tridiag}^{n_v,n_v}(1,-2,1).$$

$$Z^w := (z^w(x_i,v_j))_{\substack{i=1,\dots,n_x\\ j=1,\dots,n_v}}, \quad \Sigma^w := (\sigma^w(x_i,v_j))_{\substack{i=1,\dots,n_x\\ j=1,\dots,n_v}}, \quad u_t^{n_x,n_v} := (u_t(x_i,v_j))_{\substack{i=1,\dots,n_x\\ j=1,\dots,n_v}}$$

for $Z = F, G, H$, $z = f, g, h$, respectively, and $w \in \{x, v, xx, xv, vv\}$.

Method of Lines

$$f^x(x_i, v_j)\partial_x u_t(x_i, v_j) \approx f^x(x_i, v_j)\frac{u_t(x_{i+1}, v_j) - u_t(x_{i-1}, v_j)}{2\Delta x}$$

for all $i = 1, \ldots, n_x$ and $j = 1, \ldots, n_v$.
In our notations a derivative in x is a multiplication of the corresponding finite-difference matrix from the left to $u_t^{n_x,n_v}$, i.e.

$$\left(f^x(x_i, v_j)\frac{u_t(x_{i+1}, v_j) - u_t(x_{i-1}, v_j)}{2\Delta x} \right)_{\substack{i=1,\ldots,n_x \\ j=1,\ldots,n_v}} = F^x \odot \left(D^x \cdot u_t^{n_x,n_v} \right).$$

A derivative in v on the other hand is a multiplication from the right with the transposed matrix. To get them both on the left hand side we need to vectorize the equation.

Vectorization

Using the Hadamard or element-wise product yields

$$\operatorname{vec}\left(F^x \odot \left(D^x \cdot u_t^{n_x,n_v}\right)\right) = \operatorname{diag}\left(\operatorname{vec}\left(F^x\right)\right) \cdot \operatorname{vec}\left(D^x \cdot u_t^{n_x,n_v}\right).$$

Using the Kronecker product yields

$$\operatorname{vec}\left(D^x u_t^{n_x,n_v}\right) = \operatorname{vec}\left(D^x u_t^{n_x,n_v} I_{n_v}\right) = \left(I_{n_v} \otimes D^x\right) U_t^{n_x n_v}.$$

In total, we have

$$\left[f^x(x_i, v_j)\partial_x u_t(x_i, v_j)\right]_{\substack{i=1,\ldots,n_x \\ j=1,\ldots,n_v}} = \operatorname{diag}\left(\operatorname{vec}\left(F^x\right)\right) \cdot \left(I_{n_v} \otimes D^x\right) \cdot U_t^{n_x n_v}.$$

Vectorization

Applying this logic to all other summands in the (SPDE) yields

$$
\begin{aligned}
B := {} & \operatorname{diag}\left(\operatorname{vec}\left(H\right)\right) \\
& + \operatorname{diag}\left(\operatorname{vec}\left(F^{x}\right)\right) \cdot \left(I_{n_v} \otimes D^{x}\right) \\
& + \operatorname{diag}\left(\operatorname{vec}\left(F^{v}\right)\right) \cdot \left(D^{v} \otimes I_{n_x}\right) \\
& + \frac{1}{2}\operatorname{diag}\left(\operatorname{vec}\left(G^{xx}\right)\right) \cdot \left(I_{n_v} \otimes D^{xx}\right) \\
& + \operatorname{diag}\left(\operatorname{vec}\left(G^{xv}\right)\right) \cdot \left(D^{v} \otimes D^{x}\right) \\
& + \frac{1}{2}\operatorname{diag}\left(\operatorname{vec}\left(G^{vv}\right)\right) \cdot \left(D^{vv} \otimes I_{n_x}\right) \\
A := {} & \operatorname{diag}\left(\operatorname{vec}\left(\Sigma\right)\right) \\
& + \operatorname{diag}\left(\operatorname{vec}\left(\Sigma^{x}\right)\right) \cdot \left(I_{n_v} \otimes D^{x}\right) \\
& + \operatorname{diag}\left(\operatorname{vec}\left(\Sigma^{v}\right)\right) \cdot \left(D^{v} \otimes I_{n_x}\right).
\end{aligned}
$$

Stochastic Langevin equation

$$h \equiv f^v \equiv g^{xx} \equiv g^{xv} \equiv \sigma^x \equiv 0, \quad f_x(x,v) := -v, \quad g^{vv} \equiv a, \quad \sigma^v \equiv \sigma. \tag{3.5}$$

In this special case, there exists an explicit fundamental solution Γ for $0 < \sigma \leq \sqrt{a}$ (cf. Pascucci and Pesce (2022):p. 4 Proposition 1.1.), which is given by

$$\Gamma\left(t,z;0,\zeta\right) := \Gamma_0\left(t, z - m_t(\zeta)\right),$$
$$\Gamma_0\left(t,[x,v]\right) := \frac{\sqrt{3}}{\pi t^2(a-\sigma^2)} \exp\left(-\frac{2}{a-\sigma^2}\left(\frac{v^2}{t} - \frac{3vx}{t^2} + \frac{3x^2}{t^3}\right)\right)$$

where $\zeta := (\xi, \eta)$ is the initial point and

$$m_t(\zeta) := \begin{pmatrix} \xi + t\eta - \sigma \int_0^t W_s ds \\ \eta - \sigma W_t \end{pmatrix}.$$

Stochastic Langevin equation

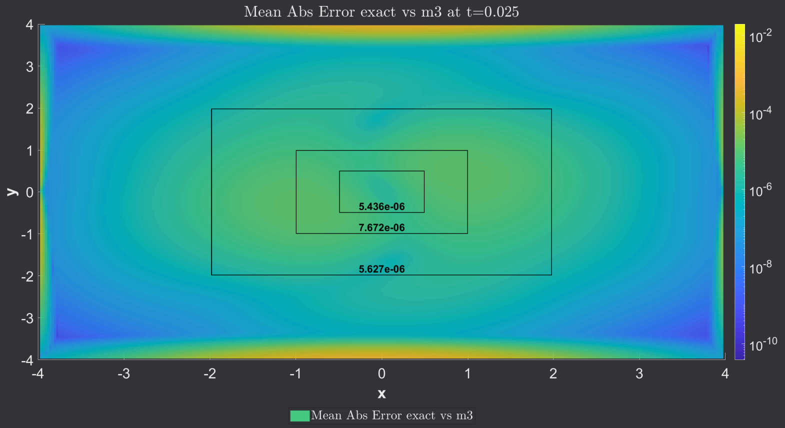
Having the fundamental solution, we can solve the Cauchy-problem by integrating against the initial datum, i.e.

$$u_t(x,v) = \int_{\mathbb{R}^2} \Gamma(t,[x,v];0,[\xi,\eta])\phi(\xi,\eta)d\xi d\eta.$$

To get an explicit solution for the double integral, we will choose ϕ to be Gaussian, i.e.

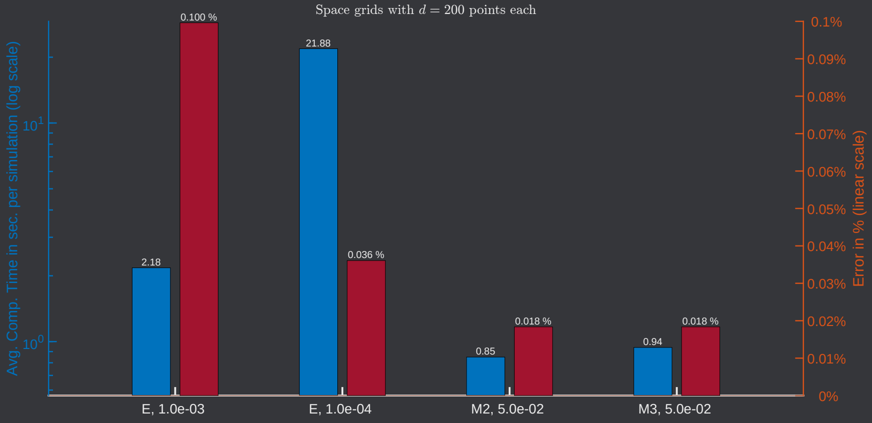
$$\phi(\xi,\eta) := \exp\left(-\frac{(\xi^2+\eta^2)}{2}\right).$$

Absolute Errors



In the case $d = 300$ and $\Delta = 2.5e-2$ on $[-4,4]\times[-4,4]$

Computational times vs Error level



Thank you for your attention!

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