Efficient Sensitivity Computations for xVA

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Monte Carlo Exposure Simulation

- Let $V(X)$ be an asset (or portfolio) over a risk-factor X_t (e.g. interest rate, stock price, ...) and let $E(t) := \max(0, V_t)$ be its positive exposure.
- The *expected exposure* at time t (as seen today at t_0) is

$$\mathrm{EE}(t_0, t) := \mathbb{E}^{\mathbb{Q}}\big[D(t_0, t)E(t)|\mathscr{F}_{t_0}\big].$$

- Standard MC simulation approach: Obtain paths of the underlyings along the time horizon

$$\begin{aligned} \text{Interest rate:} &\quad \{r_t(\omega_j)\colon t \in [t_0, T], j = 1, \dots, M\}, \\ \text{Underlying:} &\quad \{X_t(\omega_j)\colon t \in [t_0, T], j = 1, \dots, M\}, \end{aligned}$$

and compute the empirical estimator

$$\mathrm{EE}(t_0, t) \approx \frac{1}{M}\sum_{j=1}^{M} D(t_0, t; \omega_j)\max\big(0, V_t(X_t(\omega_j))\big).$$

Stochastic Collocation Monte Carlo Sampler

- Calculation of expected exposure profile ($\mathrm{EE}(t_0, t)$ for all $t \in [t_0, T]$) is expensive:
  (Number of time steps $\times$ number of paths) portfolio valuations!
- Stochastic collocation: Replace expensive portfolio valuation

$$V_t : X_t(\omega) \mapsto V_t(X_t(\omega))$$

  by polynomial approximation $g_t \approx V_t$.

  1. Evaluate N exact points: $\big(x_i, V_t(x_i)\big)$.
  2. Construct polynomial approximation g_t s.t. $g_t(x_i) = V_t(x_i)$.
  3. $\mathrm{EE}(t_0, t) \approx \frac{1}{M} \sum_{j=1}^{M} D(t_0, t; \omega_j) \max\big(0, g_t(X_t(\omega_j))\big)$.

- Stochastic Collocation Monte Carlo sampler requires only (Number of time steps $\times$ N) exact portfolio valuations.

L.A. Grzelak, J.A.S. Witteveen, M. Suárez-Taboada, C.W. Oosterlee. The Stochastic Collocation Monte Carlo Sampler. Quantitative Finance, 2019.

L.A. Grzelak. Sparse Grid Method for Highly Efficient Computation of Exposures for xVA. arXiv:2104.14319, 2021.

More general xVA (CVA with wrong-way risk)

Credit valuation adjustment (CVA) with full independence between components is "$\mathrm{CVA} = \mathrm{LGD} \times \mathrm{PD} \times \mathrm{EE}$". With correlations between exposure default (modelled with some stochastic intensity λ):

$$
\begin{aligned}
\mathrm{CVA}(t) &= \mathrm{LGD}\ \mathbb{E}^{\mathbb{Q}}\Big[D(t,t_D)\mathbb{1}_{\{t_D \le T\}} \max\big(V_{t_D},0\big)|\mathscr{F}_t\Big] \\
&= \mathrm{LGD}\ \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T D(t,s)\max\big(0,V_s\big)\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{t_D \in [s,s+\mathrm{d}s)\}}|\mathscr{F}_T]|\mathscr{F}_t\Big] \\
&= \mathrm{LGD}\ \int_t^T \mathbb{E}^{\mathbb{Q}}\Big[D(t,s)\mathrm{e}^{-\int_t^s \lambda_u \mathrm{d}u}\lambda_s \max\big(0,V_s\big)|\mathscr{F}_t\Big]\mathrm{d}s \\
&=: \mathrm{LGD}\ \int_t^T \mathbb{E}^{\mathbb{Q}}\Big[G(t,s)\max\big(0,V_s\big)|\mathscr{F}_t\Big]\mathrm{d}s.
\end{aligned}
$$

More general xVA (CVA with wrong-way risk)

$$\mathrm{CVA}(t) = \mathrm{LGD} \int_t^T \mathbb{E}^{\mathbb{Q}}\Big[G(t,s)\max\big(0,V_s\big)|\mathscr{F}_t\Big]\mathrm{d}s$$

Simulation approach:

- Simulate paths $\{\big(r_t(\omega_j),\lambda_t(\omega_j),X_t(\omega_j)\big)\colon t\in[t_0,T], j=1,\ldots,M\}$.
- $\mathbb{E}^{\mathbb{Q}}\big[G(t,s)\max(0,V_s)|F_t\big] \approx \frac{1}{M}\sum_{j=1}^{M} G(t,s;\omega_j)\; g_s(X_s(\omega_j))$.
- $G(t,s;\omega) := \exp\big(-\int_t^s (r_u(\omega)+\lambda_u(\omega))\mathrm{d}u\big)\lambda_s(\omega)$ does not require portfolio valuations.
- The stochastic collocation Monte Carlo sampler only touches the portfolio valuation; completely flexible for advanced xVA frameworks!

Sensitivities of expected exposures

1. Obtain yield curve ϕ_0 from market instruments $A_1, \ldots, A_m$.
2. Obtain shocked yield curve ϕ_i by shocking the market quote K_i of constructing instrument A_i (e.g. swap rate $+1$bp), $i = 1, \ldots, m$.
3. Simulate interest rate paths in normal and shocked market:

$$\{r_t(\omega_j)\colon t \in [t_0, T], j = 1, \ldots, M, \text{yield curve} = \phi_0\}$$
$$\{r_t^i(\omega_j)\colon t \in [t_0, T], j = 1, \ldots, M, \text{yield curve} = \phi_i\}$$

4. Compute expected exposures

$$\mathrm{EE}(t) \approx \frac{1}{M} \sum_{j=1}^{M} \exp\Big(-\int_{t_0}^{t} r_s(\omega_j)\mathrm{d}s\Big) \max\Big(0, V_t\big(r_t(\omega_j)\big)\Big),$$

$$\mathrm{EE}^i(t) \approx \frac{1}{M} \sum_{j=1}^{M} \exp\Big(-\int_{t_0}^{t} r_s^i(\omega_j)\mathrm{d}s\Big) \max\Big(0, V_t^i\big(r_t^i(\omega_j)\big)\Big).$$

5. Compute difference quotients $\frac{EE(t) - EE^i(t)}{h}$.

Sensitivities of expected exposures with collocation

$$
\begin{aligned}
\frac{\partial}{\partial K_i}\mathrm{EE}(t_0, t) &= \mathbb{E}^{\mathbb{Q}}\big[\frac{\partial}{\partial K_i}\big(D(t_0, t)\max(V_t, 0)\big)\big] \\
&= \mathbb{E}^{\mathbb{Q}}\big[\big(\frac{\partial}{\partial K_i}D(t_0, t)\big)\max(V_t, 0) + D(t_0, t)\frac{\partial \max(V_t, 0)}{\partial K_i}\big] \\
&\approx \sum_{j=1}^{M}\Big(\frac{\partial}{\partial K_i}D(t_0, t; \omega_j)\Big)\max\big(g_t(r_t(\omega_j)), 0\big) \\
&\quad + D(t_0, t; \omega_j)\frac{\max\big(g_t^i(r_t^i(\omega_j)), 0\big) - \max\big(g_t(r_t(\omega_j)), 0\big)}{\Delta K}.
\end{aligned}
$$

Can directly apply stochastic collocation method:

Standard market approximator:	$g_t \approx V_t,$
Shocked market approximator:	$g_t^i \approx V_t^i.$

$\implies$ $2N$ exact valuations at each time step (down from $2M$).

Reducing the number of exact valuations

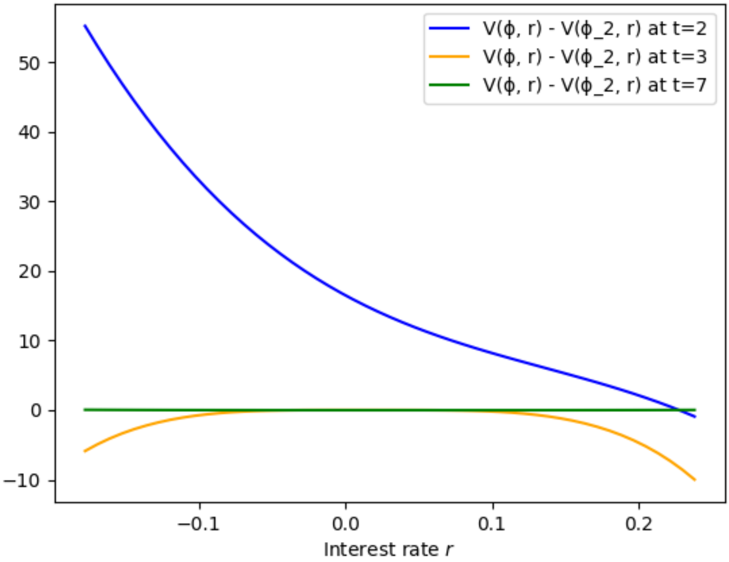
- Practitioners care about a sensitivity profile: $\frac{\mathrm{dEE}(t)}{\mathrm{d}K_i}$ for a range of market instruments A_i (used in yield curve construction) with market quotes K_i, $i = 1, \ldots, m$.
- Full collocation approach requires $N \cdot (m+1)$ exact valuations (N for V_t and N more for each V_t^i).

Idea: Difference between V_t^i and V_t may be well approximated by a polynomial of degree $d < N$:

$$h_t^i \approx V_t^i - V_t,$$

reducing the number of additional exact valuations.

Valuation differences between V_t and $V_t^{i=2}$ for $t \in \{2, 3, 7\}$



Efficient sensitivities of expected exposures with collocation

Exact valuation reduction scheme:

- Construct approximator of standard valuation $g_t \approx V_t$ based on data points $\big(r_k(t), V_t(r_k(t))\big)$.
- Construct low-degree difference approximation

$$h_t^i \approx V_t^i - g_t$$

$$h_t^i(x) := \sum_{k=1}^{d} \left(V_t^i(r_k^i) - g(r_k^i)\right) \ell_k^i(x)$$

  with only d additional exact valuations $V_t^i(r_k^i)$.
- Approximate $V_t^i \approx \widetilde{g}_t = g_t + h_t^i$.
- Requires $N + dm$ exact valuations (down from $N + Nm$)

The low-degree difference polynomial

Since V_t^i and g_t^i coincide in r_k^i, we can write

$$\begin{aligned} h_t^i(x) &:= \sum_{k=1}^{d} \left(V_t^i(r_k^i) - g(r_k^i) \right) \ell_k^i(x) \\ &= \sum_{k=1}^{d} g_t^i(r_k^i)\ell_k^i(x) - \sum_{k=1}^{d} g(r_k^i)\ell_k^i(x) \\ &=: p_t^i - p_t, \end{aligned}$$

where we have

$$\begin{aligned} p_t &\approx g_t, \\ p_t^i &\approx g_t^i. \end{aligned}$$

Uniqueness of polynomial interpolation guarantees as $d \to N$:

$$\left. \begin{aligned} p_t &\longrightarrow g_t \\ p_t^i &\longrightarrow g_t^i \end{aligned} \right\} \implies \widetilde{g}_t^i \longrightarrow g_t^i.$$

Error analysis

- Assume approximation bounds (on closed interval):

$$\|V_t - g_t\| = \varepsilon(t) \xrightarrow{N\to\infty} 0,$$
$$\|V_t^i - g_t^i\| = \varepsilon_i(t) \xrightarrow{N\to\infty} 0.$$

- Easy to obtain bounds for components p_t, p_t^i of h_t^i (target functions g_t, g_t^i are polynomials):

$$\|g_t - p_t\| =: \delta(t) \xrightarrow{d\to N} 0,$$
$$\|g_t^i - p_t^i\| =: \delta_i(t) \xrightarrow{d\to N} 0.$$

- Thus the low-degree approximation has an error of

$$\|g_t^i - \widetilde{g}_t^i\| \leq \|g_t^i - p_t^i\| + \|g_t - p_t\| = \delta(t) + \delta_i(t)$$

- And we can find an overall approximation error of

$$\|V_t^i - \widetilde{g}_t^i\| \leq \|V_t^i - g_t^i\| + \|g_t^i - \widetilde{g}_t^i\| \leq \varepsilon_i(t) + \delta_i(t) + \delta(t).$$

Error analysis

Analogously, we can compare the expected exposure sensitivities:

$$\frac{\partial \mathrm{EE}(t_0, t)}{\partial K_i} \approx \mathbb{E}_{t_0}\left[\frac{\partial D(t_0, t)}{\partial K_i} \max(V_t, 0) + D(t_0, t)\frac{V_t^i - V_t}{\Delta K}\right] =: \Psi_{\mathrm{fd}}(t),$$

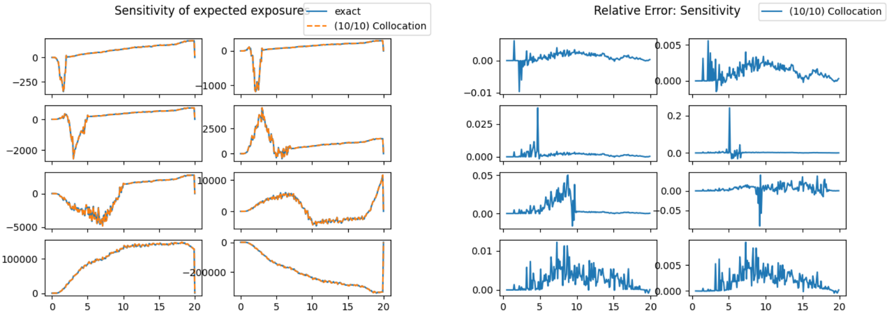
$$\frac{\partial \mathrm{EE}_{\mathrm{coll}}(t_0, t)}{\partial K_i} \approx \mathbb{E}_{t_0}\left[\frac{\partial D(t_0, t)}{\partial K_i} \max(g_t, 0) + D(t_0, t)\frac{\widetilde{g}_t^i - g_t}{\Delta K}\right] =: \Psi_{\mathrm{coll}}(t).$$

to obtain

$$\begin{aligned}|\Psi_{\mathrm{fd}}(t) - \Psi_{\mathrm{coll}}(t)| =& \varepsilon(t)\mathbb{E}_{t_0}\left[\left|\frac{\partial \exp(-\int_{t_0}^t r(s)\mathrm{d}s)}{\partial K_i}\right|\right] \\ &+ \frac{\varepsilon(t) + \varepsilon_i(t) + \delta_i(t) + \delta(t)}{\Delta K_i} P(t_0, t).\end{aligned}$$

Numerical experiment, large swap portfolio

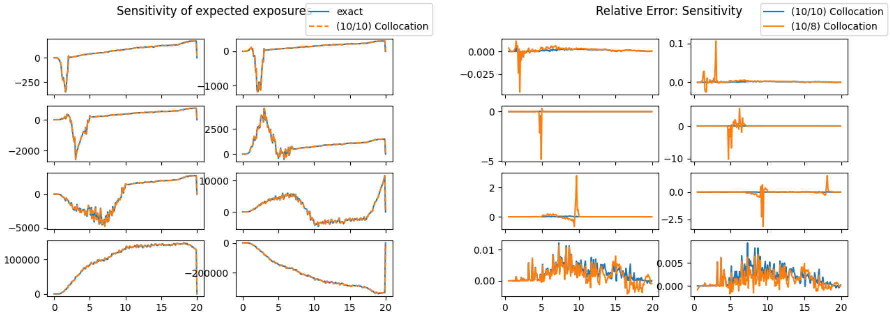
Full collocation: $d = N = 10$



(Row-wise: Sensitivity w.r.t. 1, 2, 3, 5, 7, 10, 20, 30-year instrument on yield curve)

Numerical experiment, large swap portfolio

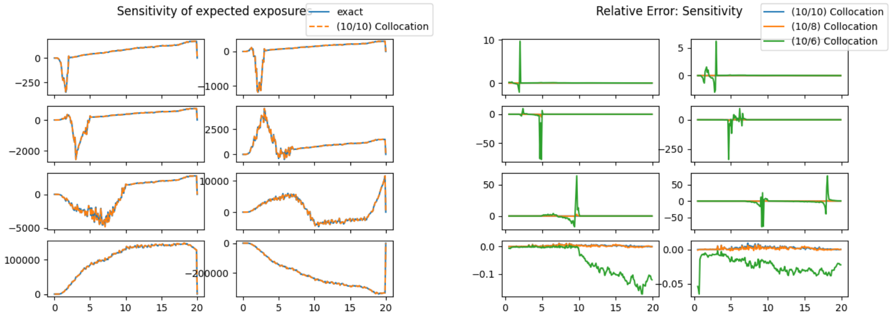
Reduction: $d = 8$, $N = 10$



(Row-wise: Sensitivity w.r.t. 1, 2, 3, 5, 7, 10, 20, 30-year instrument on yield curve)

Numerical experiment, large swap portfolio

Reduction: $d = 6, \;\; N = 10$



(Row-wise: Sensitivity w.r.t. 1, 2, 3, 5, 7, 10, 20, 30-year instrument on yield curve)

Outlook

- We have shown how to drastically reduce the number of exact portfolio valuations in xVA sensitivity.
- Success of the method relies entirely on the choice of interpolation points, particularly the d points for the difference polynomial.
- For convergence proofs we prefer Chebyshev points, in practise we rely on quadrature points.