An XVA Model in a Multi-Currency Setting
&
An ALM Model for Life Insurance Companies

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Framework

- Trade between a non-defaultable hedger (H) and a defaultable investor (I)
- Multi-ccy framework: domestic ccy D, foreign ccys $C_0, \ldots, C_N$
- Dynamics under the risk neutral probability measure of the domestic market:

Underlying assets: $dS_t^i = (r^i - q^i - \rho^{S^i,X^i}\sigma^{S^i}\sigma^{X^i})S_t^i\,dt + \sigma^{S^i}S_t^i\,dW_t^{S^i}, \quad i = 1, \ldots, N$

FX rates: $dX_t^{D,C_j} = (r^D - r^j)X_t^{D,C_j}\,dt + \sigma^{X^j}X_t^{D,C_j}\,dW^{X^j}, \quad j = 0, \ldots, N$

I's credit spread: $dh_t = (\mu^h - M^h\sigma^h)\,dt + \sigma^h\,dW_t^h$

- I's intensity of default: $\lambda = \frac{h}{1-R}$, where R is the I's recovery rate

$$\mu^h - M^h\sigma^h = -\kappa\lambda \quad \Rightarrow \quad dh_t = \frac{-\kappa}{1-R}h_t\,dt + \sigma^h\,dW_t^h$$

Framework

- $S_t = (S_t^1, \dots, S_t^N)$, $X_t = (X_t^{D,C_0}, \dots, X_t^{D,C_N})$, $\bar{X}_t = (X_t^{D,C_1}, \dots, X_t^{D,C_N})$
- I's default state at time t:

$$J_t = \begin{cases} 1 & \text{in case of default before or at time } t \\ 0 & \text{otherwise} \end{cases}$$

- Derivative value in ccy D at time t:
  - Risky: $V_t = V(t, S_t, X_t, h_t, J_t)$
  - Risk-free: $W_t = W(t, S_t, \bar{X}_t)$
- Mark-to-market derivative price: $M(t, S_t, X_t, h_t)$
- In case that I defaults:

$$V(t, S_t, X_t, h_t, 1) = RM^+(t, S_t, X_t, h_t) + M^-(t, S_t, X_t, h_t)$$

- Variation of V at default:

$$\Delta V = RM^+ + M^- - V$$

Building the replicating portfolio

Risk factors

Self-financing portfolio Π that hedges all the risk factors:

- market risk due to changes in $S^1, S^2, \ldots, S^N$
  $\Rightarrow$ fully collateralized derivatives on the same underlying assets,
  net present value H^i in ccy C_i, $H^{i,D} = H^i X^{D/C_i}$ in ccy D
- FX risk due to changes in $X^{D,C_0}, \ldots, X^{D,C_N}$
  $\Rightarrow$ FX derivatives,
  net present value E^j in ccy D
- I's spread risk due to changes in h and I's default risk
  $\Rightarrow$ two credit default swaps with different maturities written on I:
  - short term (overnight) credit default swap $CDS(t, t+dt)$
  - long term credit default swap $CDS(t, T)$

Collateral account

Collateral account C^{C_0} composed of a portfolio of bonds R^{C_0} and cash M^{C_0} (ccy C_0)

Self-financing condition of a replicating strategy

The hedger matches the spread duration of the uncollateralized part of the derivative by trading on short term bonds: $\Omega_t B(t, t+dt) = V_t - C_t^{C_0} X_t^{D/C_0}$

Building the replicating portfolio

Replicating portfolio

$$\Pi_t = \sum_{i=1}^{N} \alpha_t^i H_t^i + \sum_{j=0}^{N} \eta_t^j E_t^j + \gamma_t CDS(t,T) + \epsilon_t CDS(t,t+dt) + \Omega_t B(t,t+dt) + \beta_t$$

Bank account composition

$$\beta = -\sum_{i=1}^{N} \alpha^i H^{i,D} - \sum_{j=0}^{N} \eta^j E^j - \gamma CDS(t,T) + C^{C_0} X^{D/C_0}$$

Variation in the time interval $[t, t+dt]$:

$$d\beta_t = -\left[\sum_{i=1}^{N} \alpha_t^i (c^D + b^{D,C_j}) H_t^{i,D} + \sum_{j=0}^{N} \eta_t^j c^D E_t^j + \gamma_t c^D CDS(t,T)\right] dt$$
$$+ \left[\left(r^R + b^{D,C_0}\right) R_t^{C_0} + \left(c^D + b^{D,C_0}\right) M_t^{C_0}\right] X^{D/C_0} dt,$$

where r^R is the instantaneous repo rate associated to the bond R^{C_0}, b^{D,C_0} is the cross-ccy basis, and c^D is the OIS rate in ccy D.

Building the pricing PDE

No arbitrage + self-financing condition

$$V(t, S_t, h_t, J_t) = \Pi_t \quad \Rightarrow \quad dV_t = d\Pi_t$$

Pricing PDE for a generic mark-to-market value ($\Delta V = RM^+ + M^- - V$)

$$\frac{\partial V}{\partial t} + \mathcal{L}_{SXh} V = -\frac{h}{1-R}\Delta V + f^{H,D} V + [(r^R + b^{D,C_0} - f^{H,D})R^{C_0} + (c^D + b^{D,C_0} - f^{H,D})M^{C_0}]X^{D,C_0},$$

$$\begin{aligned}
\mathcal{L}_{SXh} = {} & \frac{1}{2}\sum_{i,k=1}^{N} \rho^{S^i S^k} \sigma^{S^i} \sigma^{S^k} S^i S^k \frac{\partial^2}{\partial S^i \partial S^k} + \frac{1}{2}\sum_{j,l=0}^{N} \rho^{X^j X^l} \sigma^{X^j} \sigma^{X^l} X^j X^l \frac{\partial^2}{\partial X^j \partial X^l} + \sum_{i=1}^{N}\sum_{j=0}^{N} \rho^{S^i X^j} \sigma^{S^i} \sigma^{X^j} S^i X^j \frac{\partial^2}{\partial S^i \partial X^j} \\
& + \frac{1}{2}(\sigma^h)^2 \frac{\partial^2}{\partial h^2} \sum_{i=1}^{N} \rho^{S^i h} \sigma^{S^i} \sigma^h S^i \frac{\partial^2}{\partial S^i \partial h} + \sum_{j=0}^{N} \rho^{X^j h} \sigma^{X^j} \sigma^h X^j \frac{\partial^2}{\partial X^j \partial h} \\
& + \sum_{i=1}^{N} (r^i - q^i - \rho^{S^i X^i} \sigma^{S^i} \sigma^{X^i}) S^i \frac{\partial}{\partial S^i} + \sum_{j=0}^{N} (r^D - r^j) X^j \frac{\partial}{\partial X^j} + (\mu^h - M^h \sigma^h)\frac{\partial}{\partial h}
\end{aligned}$$

Nonlinear ($M = V$) and linear ($M = W$) pricing PDEs

$$M = V \quad \Rightarrow \quad \frac{\partial V}{\partial t} + \mathcal{L}_{SXh} V - fV = (\bar{r} R^{C_0} + \bar{m} M^{C_0}) X^{D,C_0} + h(V)^+,$$

$$M = W \quad \Rightarrow \quad \frac{\partial V}{\partial t} + \mathcal{L}_{SXh} V - \left(\frac{h}{1-R} + f\right) V = (\bar{r} R^{C_0} + \bar{m} M^{C_0}) X^{D,C_0} + h(W)^+ - \frac{h}{1-R} W$$

PDE problems for XVA

XVA value

$U = V - W$

Final condition

$W(T, S, \bar{X}) = V(T, S, X, h) = \text{Payoff}(S, X) \quad \Rightarrow \quad U(T, S, X, h) = 0$

PDE problems

- Nonlinear final value problem ($M = V$):

$$\begin{cases} \frac{\partial U}{\partial t} + \mathcal{L}_{SXh} U - fU = h(W + U)^+ + (\bar{r} R^{C_0} + \bar{m} M^{C_0}) X^{D,C_0}, \\ U(T, S, X, h) = 0; \end{cases}$$

- Linear final value problem ($M = W$):

$$\begin{cases} \frac{\partial U}{\partial t} + \mathcal{L}_{SXh} U - \left(\frac{h}{1-R} + f \right) U = h(W)^+ + (\bar{r} R^{C_0} + \bar{m} M^{C_0}) X^{D,C_0}, \\ U(T, S, X, h) = 0. \end{cases}$$

In both cases, $(t, S, X, h) \in [0, T) \times (0, +\infty)^N \times (0, +\infty)^{N+1} \times (0, +\infty)$.

Formulation in terms of expectations

In order to compute the values of U by using the Monte Carlo method, we apply the Feynman-Kac theorem to formulate the PDE problems in terms of expectations.

Nonlinear case ($M = V$)

$$U(t,S,X,h) = E_t^Q\Big[-\int_t^T e^{-f(u-t)}\Big(h_u(W(u,S_u,\bar{X}_u) + U(u,S_u,X_u,h_u))^+ + \big(\bar{r}R_u^{C_0} + \bar{m}M_u^{C_0}\big)X_u^{D,C_0}\Big)du \,|\, S_t = S, X_t = X, h_t = h\Big]$$

Linear case ($M = W$)

$$U(t,S,X,h) = E_t^Q\Big[-\int_t^T e^{-\int_t^u(\frac{h_r}{1-R}+f)dr}\Big(h_u(W(u,S_u,\bar{X}_u))^+ + \big(\bar{r}R_u^{C_0} + \bar{m}M_u^{C_0}\big)X_u^{D,C_0}\Big)du \,|\, S_t = S, X_t = X, h_t = h\Big]$$

Numerical examples

Financial data

$r = (0.07, 0.09, 0.12)$	$\sigma^S = (0.30, 0.20)$	$q = (0.07, 0.08)$	$r^D = 0.06$
$X_0 = (0.13, 0.89, 1.12)$	$\sigma^X = (0.38, 0.40, 0.35)$	$R_0^{C_0} = 25$	$M_0^{C_0} = 25$
$h_0 = 0.20$	$\kappa = 0.01$	$\sigma^h = 0.2$	$R = 0.3$
$c^D = 0.06$	$r^R = 0.05$	$b^{D,C_0} = 0.02$	$f = 0.06$
$K^1 = 12$	$K^2 = 15$	$K = 5$	

Payoff

Best of put/put option $\Rightarrow$ $G(t, S_t^1, S_t^2, X_t^{D,C_1}, X_t^{D,C_2}) = \max((K^1 - S_t^1 X_t^{D,C_1})^+, (K^2 - S_t^2 X_t^{D,C_2})^+)$

Best of put/put option

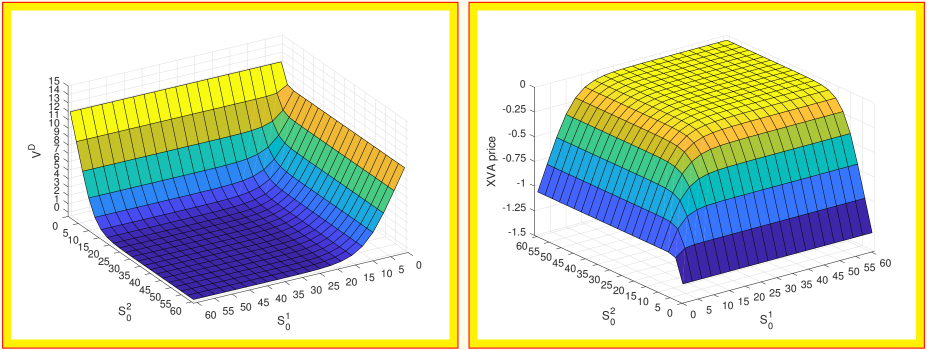


Figure: Risky price and XVA price in the nonlinear case

Best of put/put option

σ^X	XVA price
(0.275,0.05,0.05)	[-0.2039,-0.1952]
(0.275,0.05,0.50)	[-0.3027,-0.2900]
(0.275,0.50,0.05)	[-0.2898,-0.2766]
(0.275,0.50,0.50)	[-0.3710,-0.3563]
(0.000,0.00,0.00)	[-0.2000,-0.1914]

Table: XVA price confidence intervals in the nonlinear case for different sets of FX rates volatilities values

An ALM model for life insurance companies

1. General Framework
2. Portfolio Rebalancing Strategy
2. Mortality, Surrender and New Production Models
4. Future Cash Flows
5. Numerical Results

Assets and Liabilities Portfolios

Stochastic Asset Liability Management

We build a scenario-based stochastic **ALM** model with **dynamic reinvestment strategy**.

Assets Portfolio

- Bonds, divided into buckets of duration
- Equity
- Cash

Liabilities Portfolio

With-profit life policies:

- Single or periodic premiums
- Saving account growth rate: $\max(g, \beta R^P)$
- Surrender option
- New production

Model points: policies are gathered together according to:

- minimum guaranteed rate of return;
- maturity;
- age of policyholder.

First Stage of Portfolio Rebalancing Strategy

Nonlinearly constrained optimization problem

- **Objective function**: distance between asset duration and liability duration;
- **Constraint on portfolio performance**: portfolio return is near a benchmark return;
- **Typical constraints on portfolio composition**.

At period k search for an optimal array of asset classes weights $\boldsymbol{\alpha}_k$

Minimize $\quad (A^D(\boldsymbol{\alpha}_k) - L_k^D)^+, \quad \boldsymbol{\alpha}_k = (\alpha_k^{B1}, \alpha_k^{B2}, \alpha_k^{B3}, \alpha_k^{B4}, \alpha_k^{E}, \alpha_k^{C})$

Subject to
$$\begin{cases} \beta^L R_{k+1}^I \le R_{k+1}^P \le \beta^U R_{k+1}^I, \quad \text{with constant } \beta^L, \beta^U \\ \sum_{i\in I_\alpha} \alpha_k^i = 1 \text{ (budget constraint)} \\ \alpha_k^i \ge 0, \quad \forall i \in I_\alpha \text{ (no short selling constraint)} \\ \sum_{n=1}^4 \alpha_k^{Bn} \ge 0.70, \quad \alpha_k^E \le 0.20 \text{ (investement policy constraints)} \\ \left|\alpha_k^i - \alpha_{k-1}^i\right| \le 0.05, \quad \forall i \in I_\alpha \text{ (turnover constraint)} \\ \sum_{i\in I_\alpha} \left|\alpha_k^i - \alpha_{k-1}^i\right| \le 0.30 \text{ (turnover constraint)} \end{cases}$$

Second Stage of portfolio optimization

Sectorial optimization problems

- $i \in \{B1, B2, B3, B4, E\}$
- $\boldsymbol{R}^i_k$: vector of sub-sectors returns at period k for asset class i
- N_i: number of sub-sectors for asset class i
- U: utility function

At each period k search for the optimal weights vectors $\boldsymbol{\omega}^i_k = (\omega^{i,1}_k, \dots, \omega^{i,N_i}_k)$

$$\text{Maximize} \quad \mathrm{E}_k \left[\max_{\bar{\boldsymbol{\omega}}^i_{k+1}} \mathrm{E}_{k+1} \left[\max_{\bar{\boldsymbol{\omega}}^i_{k+2}} \mathrm{E}_{k+2} \left[\dots \max_{\bar{\boldsymbol{\omega}}^i_{T-1}} \mathrm{E}_{T-1} \left[\mathrm{U}(\boldsymbol{\omega}^i_{T-1} \cdot \boldsymbol{R}^i_T) \right] \dots \right] \right] \right]$$

$$\text{Subject to} \quad \sum_{j=1}^{N_i} \omega^{i,j}_k = \alpha^i_k$$

Cash Flows

Macaulay's Formula

$$L_k^D = \frac{\sum_{j>k} j d_{j|k} cf_{j|k}}{\sum_{j>k} d_{j|k} cf_{j|k}}$$

- $d_{j|k}$: price at period k of a zero-coupon bond with tenor j ($G1++$ model)
- $cf_{j|k}$: expected cash outflows at period j evaluated at period k

Death payments	$D_{k,i} = \boldsymbol{n}_{k,i}^D \cdot \boldsymbol{l}_{k,i}^D$	Premium payments	$\Pi_{k,i} = n_{k-1,i} l_{k,i}^{\Pi}$
Surrender payments	$\Gamma_{k,i} = \boldsymbol{n}_{k,i}^S \cdot \boldsymbol{l}_{k,i}^S$	New production payments	$P_{k,i} = n_{k,i}^P l_{k,i}^P$
Maturity payments	$M_{k,i} = \boldsymbol{n}_{k,i}^M \cdot \boldsymbol{l}_{k,i}^M$		

Cash outflows

$$cf_{k,i} = \begin{cases} \Gamma_{k,i} + D_{k,i} & \text{if } t_k < T_i, \\ M_{k,i} + D_{k,i} & \text{if } t_k = T_i, \\ 0 & \text{otherwise} \end{cases}$$

$$cf_k = \sum_{i=1}^{N_M} cf_{k,i}$$

Mortality, Surrender and New Production Models

Mortality Model

Number of policyholders who entered into the contract at time s and die at period k:

$$ {}_s n_{k,i}^{D,\mathcal{M}} \sim Bin({}_s n_{k-1,i}^{\mathcal{M}}, p_{k,i}^{D,\mathcal{M}}), $$

$$ {}_s n_{k,i}^{D,\mathcal{F}} \sim Bin({}_s n_{k-1,i}^{\mathcal{F}}, p_{k,i}^{D,\mathcal{F}}), $$

where $p_{k,i}^{D,\mathcal{M}}$ and $p_{k,i}^{D,\mathcal{F}}$ are given by specific life tables, depending only on age.

Surrender Model

Number of policyholders who entered into the contract at time s and surrender at period k:

$$ {}_s n_{k,i}^{S} \sim Bin({}_s n_{k-1,i}, p_{k,i}^{S}). $$

New Production Model

Number of policyholders who entered into the contract at time k:

$$ n_{k,i}^{P} \sim Bin(n_{k-1,i}, p_{k,i}^{P}). $$

Surrender and New Production Probabilities

For each model point m_i, we define

$$\delta r^S_{k,i} = (R^I_k - \max(g_{k,i}, \beta_{k,i} R^P_k))^+ \quad \text{and} \quad \delta r^P_{k,i} = (\max(g_{k,i}, \beta_{k,i} R^P_k) - R^I_k)^+,$$

where R^I_k is a benchmark rate of return at period k.

- If $\delta r^S_{k,i}$ is in the threshold interval I^q, then the surrender probability at period k is given by $p^S_{k,i} = p^S_{qk}$;
- If $\delta r^P_{k,i}$ is in the threshold interval I^q, then the new prod probability at period k is given by $p^P_{k,i} = p^P_{qk}$.

		Period				
		0	1	2	$\cdots$	$T-1$
Intervals	I^1	p^S_{10}, p^P_{10}	p^S_{11}, p^P_{11}	p^S_{12}, p^P_{12}	$\cdots$	p^S_{1T-1}, p^P_{1T-1}
	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	I^Q	p^S_{Q0}, p^P_{Q0}	p^S_{Q1}, p^P_{Q1}	p^S_{Q2}, p^P_{Q2}	$\cdots$	p^S_{QT-1}, p^P_{QT-1}

Future Cash Flows

When computing the company balance sheet projections, we consider not only future maturity and death payments, but also future surrender payments and all the cash flows due to new production.

At each time k, for each model point i and for $j > k$, we have to compute:

- $E[\max(g_{j,i}, \beta_{j,i} R_j^P)|\mathcal{F}_k]$
- $E[\delta r_{j,i}^S|\mathcal{F}_k] = E[(R_j^I - \max(g_{j,i}, \beta_{j,i} R_j^P))^+|\mathcal{F}_k]$
- $E[\delta r_{j,i}^P|\mathcal{F}_k] = E[(\max(g_{j,i}, \beta_{j,i} R_j^P) - R_j^I)^+|\mathcal{F}_k]$

$\Rightarrow$ Least Squares Monte Carlo method

Numerical Results

Assumptions

- All contracts have the same value, say €10 000, in the moment they are signed.
- All policies expire at the same future date, say at time $T = 10$ years.
- At time 0 policies are equally distributed between male and female policyholders (**gender equality**).
- Portfolio is rebalanced at each time step (**one year**).
- Policyholders pay a **single premium** at the beginning of the contract.
- The participation rate is the same for all model points and is constant over time ($\beta = 95\%$).
- $L_0 = 88.7\% A_0$.

Initial scenario

Asset class		Weight
$B1$	bonds, maturity 1-3	21.09%
$B2$	bonds, maturity 3-5	22.91%
$B3$	bonds, maturity 5-10	35.79%
$B4$	bonds, maturity >10	15.38%
E	equity	3.74%
C	cash	1.09%

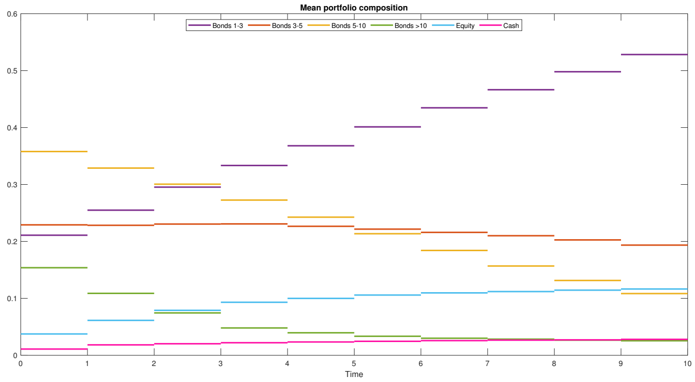
Table: Portfolio composition

	Minimum guarantee		
Age	0%	1%	2%
[40, 44]	50	5	1
[45, 49]	55	5	3
[50, 54]	55	10	3
[55, 59]	60	25	15
[60, 64]	70	80	23
[65, 69]	60	100	50

Table: Number of policies in each model point

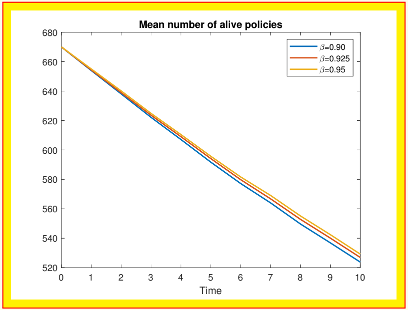
Numerical Results

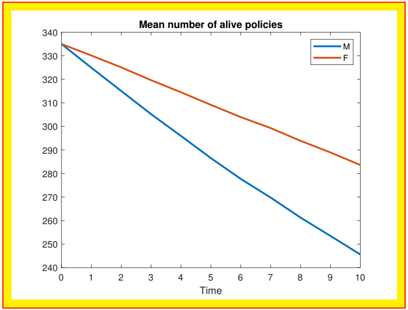
Portfolio Composition Rebalancing



Numerical Results

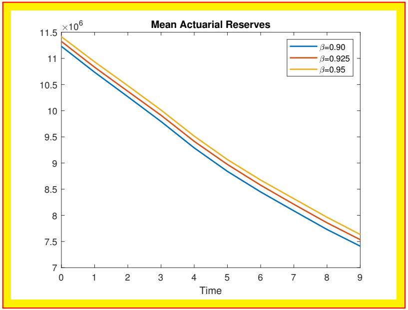
Number of alive policies

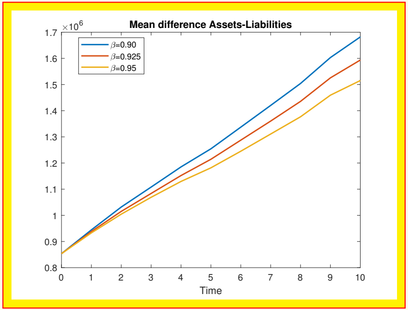




Numerical Results

Participation Rate Sensitivity





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